

OXFORD IB DIPLOMA PROGRAMME



VISIT...

ANSWERS

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MATHEMATICS HIGHER LEVEL:
CALCULUS

COURSE COMPANION

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1

Patterns to infinity

Skills check

$$1 \text{ a } \frac{3}{4n+1} - \frac{3}{4n-1} = \frac{3(4n-1) - 3(4n+1)}{(4n+1)(4n-1)} \\ = \frac{12n-3-12n-3}{16n^2-1} = \frac{-6}{16n^2-1}$$

$$\text{b } \frac{n-1}{n+2} - \frac{n}{n+3} = \frac{(n-1)(n+3) - n(n+2)}{(n+2)(n+3)} \\ = \frac{n^2-n+3n-3-n^2-2n}{n^2+2n+3n+6} \\ = -\frac{3}{n^2+5n+6}$$

$$2 \text{ a } |3x-1| < 1 \Rightarrow -1 < 3x-1 < 1 \\ \Rightarrow 0 < 3x < 2 \Rightarrow 0 < x < \frac{2}{3}$$

$$\text{b } |2x-3| \geq 4 \Rightarrow 2x-3 \geq 4 \text{ or } 2x-3 \leq -4 \\ \Rightarrow 2x \geq 7 \text{ or } 2x \leq -1 \\ \Rightarrow x \geq \frac{7}{2} \text{ or } x \leq -\frac{1}{2}$$

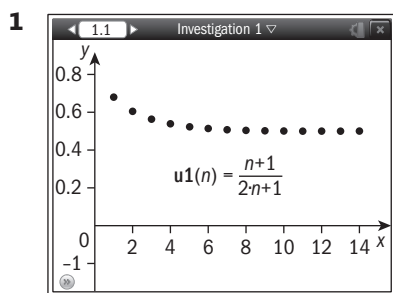
$$3 \text{ a } \frac{2n}{1+\sqrt{n}} = \frac{2n(1-\sqrt{n})}{(1+\sqrt{n})(1-\sqrt{n})} = \frac{2n(\sqrt{n}-1)}{n-1}$$

$$\text{b } \frac{n-1}{\sqrt{n}+\sqrt{n+1}} = \frac{(n-1)(\sqrt{n}-\sqrt{n+1})}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n}-\sqrt{n+1})} \\ = \frac{(n-1)(\sqrt{n}-\sqrt{n+1})}{n-n-1} \\ = (n-1)(\sqrt{n+1}-\sqrt{n})$$

$$4 \text{ a } \frac{(-1)^{n+1}}{2^n} \quad \text{b) } \frac{(-1)^{n+1}}{n+1} \quad \text{c) } \frac{2n-1}{2n}$$

$$\text{d } (-1)^{n+1} \sqrt{\frac{2n+1}{2^n}}, n \in \mathbb{Z}^+$$

Investigation 1



2 As n increases the terms get very close to $\frac{1}{2}$.

$$3 \left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \frac{1}{10} \Rightarrow \left| \frac{2n+2-2n-1}{2(2n+1)} \right| < \frac{1}{10} \\ \Rightarrow \frac{1}{2n+1} < \frac{1}{5} \\ \Rightarrow n > 2 \therefore m = 3$$

$$4 \left| \frac{n+1}{2n+1} - \frac{1}{2} \right| < \varepsilon \Rightarrow \left| \frac{2n+2-2n-1}{2(2n+1)} \right| < \varepsilon \\ \Rightarrow \frac{1}{2n+1} < 2\varepsilon \Rightarrow n > \frac{1}{2} \left(\frac{1}{2\varepsilon} - 1 \right) \\ \therefore \varepsilon = 0.01 \Rightarrow m = 25; \varepsilon = 0.001 \Rightarrow m = 250; \\ \varepsilon = 0.0001 \Rightarrow m = 2500$$

$$5 \left| \frac{n+1}{2n+1} - \frac{2}{5} \right| < \frac{1}{10} \Rightarrow \left| \frac{2n+2-4n-2}{2(2n+1)} \right| < \frac{1}{10} \\ \Rightarrow \frac{2n}{2n+1} < \frac{1}{5} \\ \Rightarrow n < \frac{1}{8}$$

No as this implies that $n < \frac{1}{8}$.

6 As n increases the terms get very close to 0.

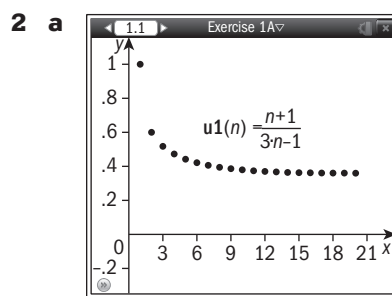
$$7 \left| \left(-\frac{1}{3} \right)^n \right| < \varepsilon \Rightarrow \frac{1}{3^n} < \varepsilon \Rightarrow n > \log_3 \left(\frac{1}{\varepsilon} \right) \\ \therefore \varepsilon = 0.01 \Rightarrow m = 6; \varepsilon = 0.001 \Rightarrow m = 7; \\ \varepsilon = 0.0001 \Rightarrow m = 9$$

8 $|u_n - L|$ can take arbitrary small values as we consider just terms after an order m (as we increase this order the terms are closer to L).

Exercise 1A

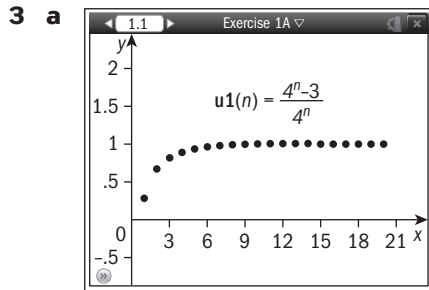
$$1 \left| \frac{n+3}{2n+1} - \frac{1}{2} \right| < \frac{1}{1000} \Rightarrow \left| \frac{2n+6-2n-1}{2(2n+1)} \right| < \frac{1}{1000} \\ \Rightarrow \frac{5}{2n+1} < \frac{1}{500} \\ \Rightarrow n > 1249.5$$

$$\therefore m = 1250$$



$$\begin{aligned} \text{b } \left| \frac{n+1}{3n-1} - \frac{1}{3} \right| &< \frac{1}{1000} \Rightarrow \left| \frac{3n+3-3n+1}{3(3n-1)} \right| < \frac{1}{1000} \\ &\Rightarrow \frac{4}{9n-3} < \frac{1}{1000} \\ &\Rightarrow n > \frac{4003}{9} \end{aligned}$$

$$\therefore m = 445$$



$$\begin{aligned} \text{b } \left| \frac{4^n - 3}{4^n} - 1 \right| &< \frac{5}{10000} \Rightarrow \left| \frac{4^n - 3 - 4^n}{4^n} \right| < \frac{1}{2000} \\ &\Rightarrow \frac{3}{4^n} < \frac{1}{2000} \\ &\Rightarrow n > \log_4 6000 \end{aligned}$$

$$\therefore m = 7$$

- 4 a Divergent as the terms of even order approach $\frac{3}{4}$ and the terms of odd order approach $-\frac{3}{4}$.
- b Convergent to zero as the absolute value of the terms get close to zero quickly.
- c Divergent as while the terms of even order get close to zero, the terms of odd order increase without bound.
- d Converges to zero as the square roots of consecutive integers are approximately equal for large values of n .

	A _n	B _n	C _n
		$=\sqrt{('n)}$	$=\sqrt{('n+1)}$
1	1000	31.6228	31.6228
2	10000	100	100.005
3	100000	316.228	316.229
4	1000000	1000	1000.
5	10000000	3162.28	3162.28

- e Diverges to positive infinity as the terms increase without bound.
- f Diverges as the terms follow the cyclic pattern 1, 0, -1, 0, 1, 0, -1, 0, ...
- g Diverges as the terms of even order are equal to 1 and the terms of odd order get close to zero.
- h Converges to $\frac{1}{2}$ as $\frac{n^3 + n + 1}{2n^3 + 3} = \frac{1}{2} \left(1 + \frac{n-2}{2n^3 + 3} \right)$ and $\frac{n-2}{2n^3 + 3}$ approaches zero as n increases.

$$\begin{aligned} 5 \quad \frac{n \cos^2(na)}{n^2 + 3} &= \underbrace{\frac{n}{n^2 + 3}}_{\text{converges to zero}} \cdot \underbrace{\cos^2(na)}_{\text{bounded}} \text{ as } 0 \leq \cos^2(na) \leq 1. \\ \text{Therefore } \lim_{n \rightarrow \infty} \frac{n \cos^2(na)}{n^2 + 3} &= 0, a \in \mathbb{R}. \text{ QED} \end{aligned}$$

Exercise 1B

$$\begin{aligned} 1 \quad \text{a } -2 \cdot \left(\frac{3}{5} \right)^n &\leq 2 \cdot (-1)^{n-1} \cdot \left(\frac{3}{5} \right)^n \leq 2 \cdot \left(\frac{3}{5} \right)^n; \text{ as} \\ \lim_{n \rightarrow \infty} \left(-2 \cdot \left(\frac{3}{5} \right)^n \right) &= \lim_{n \rightarrow \infty} 2 \cdot \left(\frac{3}{5} \right)^n = 0, \text{ by the} \\ \text{squeeze theorem } \lim_{n \rightarrow \infty} 2 \cdot (-1)^{n-1} \cdot \left(\frac{3}{5} \right)^n &= 0. \end{aligned}$$

$$\begin{aligned} 2 \quad \text{a } -1 &\leq \sin(2n) \leq 1, \text{ for all } n \in \mathbb{Z}^+ \\ \therefore 3n-1 &\leq 3n + \sin(2n) \leq 3n+1 \\ \Rightarrow \frac{3n-1}{4n-3} &\leq \frac{3n + \sin(2n)}{4n-3} \leq \frac{3n+1}{4n-3} \\ \text{as } 4n-3 > 0 & \\ \text{for all } n \in \mathbb{Z}^+; \text{ as } \lim_{n \rightarrow \infty} \frac{3n-1}{4n-3} &= \frac{3}{4} \end{aligned}$$

$$\text{b } \lim_{n \rightarrow \infty} \frac{3n-1}{4n-3} = \lim_{n \rightarrow \infty} \frac{3 - \frac{1}{n}}{4 - \frac{3}{n}} = \frac{3-0}{4-0} = \frac{3}{4} \text{ and}$$

$$\lim_{n \rightarrow \infty} \frac{3n+1}{4n-3} = \lim_{n \rightarrow \infty} \frac{3 + \frac{1}{n}}{4 - \frac{3}{n}} = \frac{3+0}{4-0} = \frac{3}{4}$$

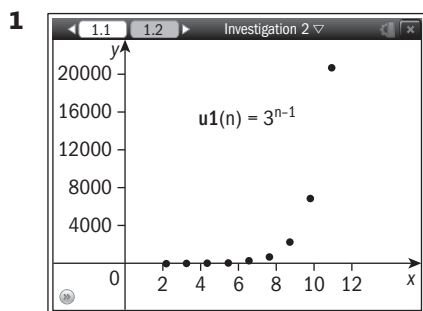
$$\begin{aligned} \text{c } \text{As } \frac{3n-1}{4n-3} &\leq \frac{3n + \sin(2n)}{4n-3} \leq \frac{3n+1}{4n-3} \text{ and} \\ \lim_{n \rightarrow \infty} \frac{3n-1}{4n-3} &= \lim_{n \rightarrow \infty} \frac{3n+1}{4n-3} = \frac{3}{4} \text{ by the squeeze} \\ \text{theorem } \lim_{n \rightarrow \infty} \frac{3n + \sin(2n)}{4n-3} &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} 3 \quad \text{a } \frac{2n}{3n+1} &\geq \frac{1}{2} \Rightarrow 4n \geq 3n+1 \Rightarrow n \geq 1 \text{ which is true} \\ \text{for all } n \in \mathbb{Z}^+; \frac{2n}{3n+1} &< \frac{2}{3} \Rightarrow 6n < 6n+2 \Rightarrow 0 < 2 \\ \text{which is true for all } n \in \mathbb{Z}^+. \\ \text{Therefore, } \frac{1}{2} &\leq \frac{2n}{3n+1} < \frac{2}{3} \text{ for all } n \in \mathbb{Z}^+. \end{aligned}$$

$$\begin{aligned} \text{b } \text{As } \frac{1}{2} &\leq \frac{2n}{3n+1} < \frac{2}{3} \text{ for all } n \in \mathbb{Z}^+ \text{ and} \\ \lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n &= \lim_{n \rightarrow \infty} \left(\frac{3}{4} \right)^n = 0 \text{ by the squeeze theorem,} \\ \lim_{n \rightarrow \infty} \left(\frac{2n}{3n+1} \right)^n &= 0 \end{aligned}$$

$$\begin{aligned} 4 \quad \text{Assumption: the sequence } \{a_n\} &\text{ converges to} \\ \text{a real number } a. \text{ As } \lim_{n \rightarrow \infty} a_{n+1} &= \lim_{n \rightarrow \infty} a_n = a, \text{ then} \\ \lim_{n \rightarrow \infty} a_{n+1} &= \lim_{n \rightarrow \infty} \frac{a_n + 2}{3} \Rightarrow a = \frac{a+2}{3} \Rightarrow a = 1 \end{aligned}$$

Investigation 2



2 The terms of the sequence increase without upper bound.

3 $3^{n-1} > 100 \Rightarrow n > 5.19... \Rightarrow m = 6$

4 $3^{n-1} \geq 1000 \Rightarrow n \geq 7.28... \Rightarrow m = 8,$

$3^{n-1} \geq 10000 \Rightarrow n \geq 9.38... \Rightarrow m = 10$ and

$3^{n-1} \geq 100000 \Rightarrow n \geq 11.47... \Rightarrow m = 12$

$\log_3(100)+1$	5.19181
$\log_3(1000)+1$	7.28771
$\log_3(10000)+1$	9.38361
$\log_3(100000)+1$	11.4795

5 $-2^n \leq 1000 \Rightarrow n > 9.96... \Rightarrow m = 10,$

$-2^n \leq 10000 \Rightarrow n > 13.2... \Rightarrow m = 14$ and

$-2^n \leq 1000000 \Rightarrow n > 16.6... \Rightarrow m = 17$

$\log_2(1000)$	9.96578
$\log_2(10000)$	13.2877
$\log_2(100000)$	16.6096

6 The terms of the sequence decrease without lower bound.

7

$\log_4(1000)$	4.98289
$\log_4(10000)$	6.64386
$\log_4(100000)$	8.30482

$|-4^n| \geq 1000 \Rightarrow n > 4.98... \Rightarrow m = 5,$

$|-4^n| \geq 10000 \Rightarrow n > 6.64... \Rightarrow m = 7$ and

$|-4^n| \geq 100000 \Rightarrow n > 8.30... \Rightarrow m = 9$

8 The absolute value of the terms of the sequence increase without upper bound.

Investigation 3

a

n	$=n/2^n$
1	0.5
10	0.00976563
100	7.8886091E-29
500	1.5274682E-148
1000	9.3326362E-299
B5	=9.3326361850171E-299

$\therefore \lim_{n \rightarrow \infty} a_n = 0$

b Explore the limit for different values of k and b . For example:

n	$=n^2/3^n$	$=n^2/4^n$
1	0.33333333	0.25
10	0.00169351	0.00009537
100	1.9403252E-44	6.2230153E-57
500	6.8756324E-23...	2.333159E-296
1000	7.563891E-47...	8.7098098E-597
C5	=8.7098098162032E-597	

n	$=(n)/3^n$	$=n/4^n$
1	0.33333333	0.25
10	0.00016935	0.00000954
100	1.9403252E-46	6.2230153E-59
500	1.3751265E-23...	4.6663181E-299
1000	7.5638913E-47...	8.7098098E-600
C5	=8.7098098162032E-600	

$\therefore \lim_{n \rightarrow \infty} b_n = 0$

c

n	$=\text{root}(3,n)$
100	1.01105
1000	1.0011
10000	1.00011
100000	1.00001
B4	=1.0000109861832

$\therefore \lim_{n \rightarrow \infty} c_n = 1$

d

n	$=\text{root}(n,n)$
100	1.00693
1000	1.00092
10000	1.00012
100000	1.00001
1000000	1.
B5	=1.0000016118109

$\therefore \lim_{n \rightarrow \infty} d_n = 1$

e $\lim_{n \rightarrow \infty} e_n = +\infty$

You need to use a powerful tool to calculate this one, e.g., Mathematica or its free version Wolfram Alpha.

Exercise 1C

$$1 \text{ a } \lim_{n \rightarrow \infty} \frac{5n^3 - n}{2n^3 + 6n^2 - 1} = \lim_{n \rightarrow \infty} \frac{5 - \frac{1}{n^2}}{2 + \frac{6}{n} - \frac{1}{n^3}} = \frac{5 - 0}{2 + 0 - 0} = \frac{5}{2}$$

$$b \lim_{n \rightarrow \infty} \frac{n^2 + 5n}{n^3 + 2n + 3} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{5}{n^2}}{1 + \frac{2}{n^2} + \frac{3}{n^3}} = \frac{0 + 0}{1 + 0 + 0} = 0$$

$$c \lim_{n \rightarrow \infty} \frac{n^2 + 6n - 1}{n + 1} = \lim_{n \rightarrow \infty} \frac{1 + \frac{6}{n} - \frac{1}{n^2}}{\frac{1}{n} + \frac{1}{n^2}} = \frac{1 + 0 - 0}{0 + 0} = +\infty$$

$$d \lim_{n \rightarrow \infty} (n^3 - n) = \lim_{n \rightarrow \infty} (n(n^2 - 1)) = (+\infty) \times (+\infty) = +\infty$$

$$e \lim_{n \rightarrow \infty} (\sqrt{n+5} - \sqrt{n+2})$$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n+5} - \sqrt{n+2})(\sqrt{n+5} + \sqrt{n+2})}{\sqrt{n+5} + \sqrt{n+2}}$$

$$= \lim_{n \rightarrow \infty} \frac{3}{\sqrt{n+5} + \sqrt{n+2}} = \frac{3}{+\infty} = 0$$

$$f \lim_{n \rightarrow \infty} \sqrt{\frac{n+5}{n+2}} = \sqrt{\lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n}}{1 + \frac{2}{n}}} = \sqrt{\frac{1+0}{1+0}} = 1$$

$$g \lim_{n \rightarrow \infty} \frac{3^n + 2^n}{4^n + 5 \cdot 3^n} = \lim_{n \rightarrow \infty} \frac{1 + \left(\frac{2}{3}\right)^n}{\left(\frac{4}{3}\right)^n + 5} = \frac{1 + 0}{+\infty + 5} = 0$$

$$h \lim_{n \rightarrow \infty} \frac{(-3)^{n+1} + 7^n}{4^{n-1} + e^n} = \lim_{n \rightarrow \infty} \frac{\left(-\frac{3}{7}\right)^n + 1}{\frac{1}{4} \cdot \left(\frac{4}{7}\right)^n + \left(\frac{e}{7}\right)^n} = \frac{0 + 1}{0 + 0} = +\infty$$

$$i \lim_{n \rightarrow \infty} \frac{\sqrt{n} - \sqrt[3]{n}}{\sqrt{n}} = \lim_{n \rightarrow \infty} \left(1 - \sqrt[6]{\frac{1}{n}}\right) = 1 - 0 = 1$$

$$j \lim_{n \rightarrow \infty} \left(\frac{n^3}{n+1} - \frac{n^3+1}{n} \right) = \lim_{n \rightarrow \infty} \frac{n^4 - n^4 - n^3 - n - 1}{n^2 + n}$$

$$= \lim_{n \rightarrow \infty} \frac{-1 - \frac{1}{n^2} - \frac{1}{n^3}}{\frac{1}{n} + \frac{1}{n^2}}$$

$$= \frac{-1 - 0 - 0}{0 + 0} = -\infty$$

$$k \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n (2k-1)}{2n^2+1} = \lim_{n \rightarrow \infty} \frac{\frac{2n-2}{2} \cdot n}{2n^2+1} = \lim_{n \rightarrow \infty} \frac{n^2 - n}{2n^2+1}$$

$$= \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n}}{2 + \frac{1}{n^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

$$l \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n (2^{k-1})}{3^n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} \frac{2^n - 1}{2 - 1}}{3^n} = \lim_{n \rightarrow \infty} \frac{2^n - 1}{2 \cdot 3^n}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{2}{3}\right)^n - \frac{1}{3^n}}{2} = \frac{0 - 0}{2} = 0$$

$$2 \text{ a } \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 \Rightarrow \lim_{n \rightarrow \infty} \ln(\sqrt[n]{n}) = \ln 1 = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \ln n = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$\text{As } \frac{\ln n}{n} \leq \frac{\ln(n+k)}{n} \leq \frac{\ln n + k}{n} \text{ and}$$

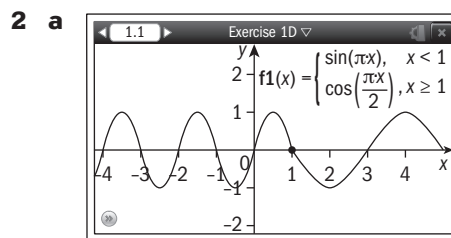
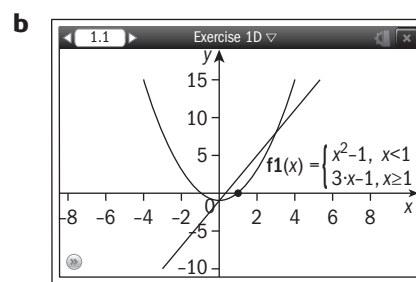
$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{n \rightarrow \infty} \frac{\ln n + k}{n} = 0, \text{ by the squeeze}$$

$$\text{theorem } \lim_{n \rightarrow \infty} \frac{\ln(n+k)}{n} = 0$$

$$b \lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \left(\frac{5}{n}\right)^n = \lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \lim_{n \rightarrow \infty} \frac{5}{n} = 1 \times 0 = 0$$

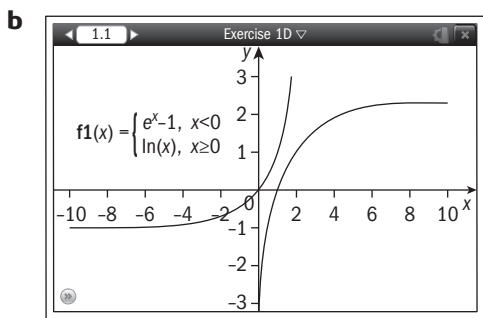
Exercise 1D

$$1 \text{ a } \lim_{x \rightarrow 1^-} (x^2 - 1) = 1 - 1 = 0 \neq \lim_{x \rightarrow 1^+} (3x - 1) = 3 - 1 = 2$$



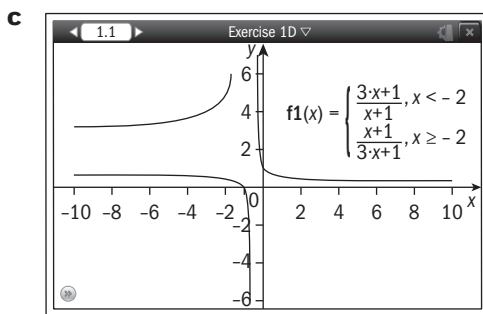
$$\lim_{x \rightarrow 1^-} \sin(\pi x) = \lim_{x \rightarrow 1^+} \cos\left(\frac{\pi x}{2}\right) = 0$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 0$$



$$\lim_{x \rightarrow 0^-} (e^x - 1) = 1 - 1 = 0 \neq \lim_{x \rightarrow 1^+} \ln(x) = -\infty$$

$\therefore \lim_{x \rightarrow 0} g(x)$ does not exist



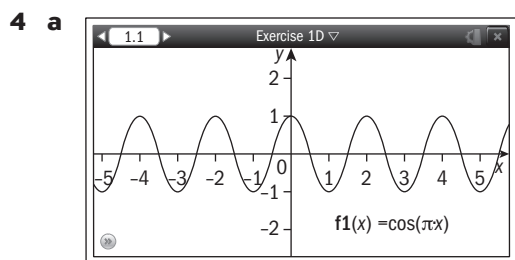
$$\lim_{x \rightarrow -2^-} \frac{3x+1}{x+1} = \frac{-5}{-1} = 5 \neq \lim_{x \rightarrow -2^+} \frac{x+1}{3x+1} = \frac{-1}{-5} = \frac{1}{5}$$

$\therefore \lim_{x \rightarrow -2} h(x)$ does not exist.

3 a $\lim_{x \rightarrow +\infty} \frac{5x-1}{2x^2+3x-1} = \lim_{x \rightarrow +\infty} \frac{\frac{5}{x} - \frac{1}{x^2}}{2 + \frac{3}{x} - \frac{1}{x^2}} = \frac{0-0}{2+0+0} = 0$

b $\lim_{x \rightarrow +\infty} \frac{5x^2-x+1}{x^2-3x+1} = \lim_{x \rightarrow +\infty} \frac{5 - \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{3}{x} + \frac{1}{x^2}} = \frac{5-0+0}{1-0+0} = 5$

c $\lim_{x \rightarrow +\infty} (\sqrt{2x+1} - \sqrt{x+1})$
 $= \lim_{x \rightarrow +\infty} \frac{(\sqrt{2x+1} - \sqrt{x+1})(\sqrt{2x+1} + \sqrt{x+1})}{\sqrt{2x+1} + \sqrt{x+1}}$
 $= \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{2x+1} + \sqrt{x+1}} = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{\frac{2}{x} + \frac{1}{x^2}} + \sqrt{\frac{1}{x} + \frac{1}{x^2}}} = 0$



b Let $a_n = 2n$ and $b_n = 2n + 1$.

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = +\infty \text{ and}$$

$$\lim_{n \rightarrow \infty} f(a_n) = 1 \neq \lim_{n \rightarrow \infty} f(b_n) = -1. \text{ Therefore}$$

$$\lim_{x \rightarrow +\infty} f(x) \text{ cannot exist.}$$

Review exercise

1 a $\lim_{n \rightarrow \infty} \frac{n^3}{2n^3-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n^3}} = \frac{1}{2-0} = \frac{1}{2}$

b $\lim_{n \rightarrow \infty} \frac{n^3}{2^n} = 0$ **c** $\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$

d $\lim_{n \rightarrow \infty} (\sqrt{n+3} - \sqrt{n}) = \lim_{n \rightarrow \infty} \frac{(\sqrt{n+3} - \sqrt{n})(\sqrt{n+3} + \sqrt{n})}{\sqrt{n+3} + \sqrt{n}}$
 $= \lim_{n \rightarrow \infty} \frac{3}{\sqrt{n+3} + \sqrt{n}} = \frac{3}{+\infty} = 0$

e $\lim_{n \rightarrow \infty} \frac{1+2+3+\dots+2n}{n^2} = \lim_{n \rightarrow \infty} \frac{n+2n^2}{n^2}$
 $= \lim_{n \rightarrow \infty} \left(\frac{1}{n} + 2\right) = 0 + 2 = 2$

f $\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n 2 \cdot 3^k}{5^n} = \lim_{n \rightarrow \infty} \frac{2 \cdot \frac{3^{n+1}-1}{3-1}}{5^n} = \lim_{n \rightarrow \infty} \frac{3^{n+1}-1}{5^n}$
 $= \lim_{n \rightarrow \infty} \left(3 \cdot \left(\frac{3}{5}\right)^n - \frac{1}{5^n}\right) = 3 \times 0 - 0 = 0$

2 e.g. $u_n = -\frac{1}{n}$

3 $\frac{3}{n^2+n} \cdot (n+1) \leq \sum_{k=0}^n \frac{3}{n^2+k} \leq \frac{3}{n^2} \cdot (n+1)$ for all
 $n \in \mathbb{Z}^+$ as $\lim_{n \rightarrow \infty} \frac{3}{n^2+n} \cdot (n+1) = \lim_{n \rightarrow \infty} \frac{3}{n^2} \cdot (n+1) = 0$ by

the squeeze theorem $\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{3}{n^2+k} = 0$

4 $\frac{n(n-1)}{4n^2+3}$
 $= \sum_{k=1}^n \frac{n-1}{4n^2+3} \leq \sum_{k=1}^n \frac{n+\cos(k\pi)}{4n^2+3} \leq \sum_{k=1}^n \frac{n+1}{4n^2+3}$
 $= \frac{n(n+1)}{4n^2+3}$ for all $n \in \mathbb{Z}^+$;

as $\lim_{n \rightarrow \infty} \frac{n(n-1)}{4n^2+3} = \lim_{n \rightarrow \infty} \frac{1 - \frac{1}{n}}{4 + \frac{3}{n^2}} = \frac{1}{4}$ and

$\lim_{n \rightarrow \infty} \frac{n(n+1)}{4n^2+3} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{4 + \frac{3}{n^2}} = \frac{1}{4}$ by the squeeze

theorem $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n+\cos(k\pi)}{4n^2+3} = \frac{1}{4}$ which is a real number. Therefore the sequence converges.

5 a $\frac{1}{3n(3n-1)}, n \in \mathbb{Z}^+$

b 1

c 0

6 $\frac{n!}{(n+2)!} \leq \frac{n! + \sqrt{n}}{(n+2)!} \leq \frac{2n!}{(n+2)!}$ for all $n \in \mathbb{Z}^+$;

$$\begin{aligned} \text{as } \lim_{n \rightarrow \infty} \frac{n!}{(n+2)!} &= \lim_{n \rightarrow \infty} \frac{n!}{(n+2)(n+1)n!} \\ &= \lim_{n \rightarrow \infty} \frac{1}{(n+2)(n+1)} = 0 \\ \text{and } \lim_{n \rightarrow \infty} \frac{2n!}{(n+2)!} &= \lim_{n \rightarrow \infty} \frac{2n!}{(n+2)(n+1)n!} \\ &= \lim_{n \rightarrow \infty} \frac{2}{(n+2)(n+1)} = 0 \end{aligned}$$

by the squeeze theorem $\lim_{n \rightarrow \infty} \frac{n! + \sqrt{n}}{(n+2)!} = 0$

- 7 a $a_1 = 3 > 0$ and each term is obtained from the previous one by adding 1 unit and then dividing the sum by 2. Therefore all the terms remain positive.

Note: you can also show it using mathematical induction.

b Let $P(n): a_{n+1} - a_n < 0$.

Step 1: $P(1): a_2 - a_1 < 0$ which is true as

$$a_2 = \frac{3+1}{2} = 2 \text{ and } a_2 - a_1 = 2 - 3 = -1 < 0$$

Step 2: Assume $P(k): a_{k+1} - a_k < 0$ true and

consider $P(k+1): a_{k+2} - a_{k+1} < 0$.

$$\text{As } a_{k+2} - a_{k+1} = \frac{a_{k+1} + 1}{2} - \frac{a_k + 1}{2} = \frac{1}{2}(a_{k+1} - a_k) < 0.$$

Therefore, $P(k)\text{true} \Rightarrow P(k+1)\text{true}$.

By the PMI, as $P(1)\text{true}$ and

$P(k)\text{true} \Rightarrow P(k+1)\text{true}$, we can conclude that

$P(n)\text{true}$ for all $n \in \mathbb{Z}^+$.

c $a_n = \left(\frac{1}{2}\right)^{n-2} + 1$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{2}\right)^{n-2} + 1\right) = 1$

d $\lim_{n \rightarrow \infty} (a_n)^n = \lim_{n \rightarrow \infty} \left(\left(\frac{1}{2}\right)^{n-2} + 1\right)^n$

$$= \lim_{n \rightarrow \infty} \underbrace{\left(\left(\frac{1}{2}\right)^{n(n-2)} + n \cdot \left(\frac{1}{2}\right)^{(n-1)(n-2)} + \dots + 1\right)}_{u_n}$$

(using binomial expansion).

$$\text{As } 1 \leq u_n \leq \left(\frac{1}{2}\right)^{n(n-2)} \sum_{k=0}^{n-1} \binom{n}{k} + 1 = \frac{2^n - 1}{2^n} \left(\frac{1}{2}\right)^{n-2} + 1$$

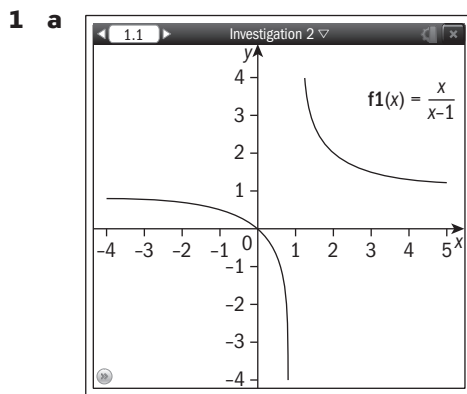
and $\frac{2^n - 1}{2^n} \left(\frac{1}{2}\right)^{n-2} + 1$, by the squeeze theorem

$$\lim_{n \rightarrow \infty} (a_n)^n = 1.$$

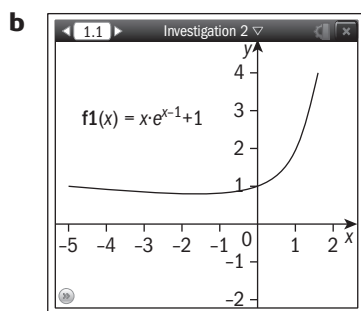
2

Smoothness in mathematics

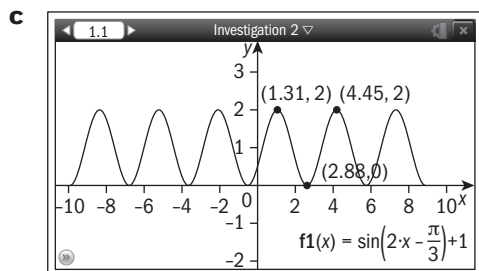
'Before you start'



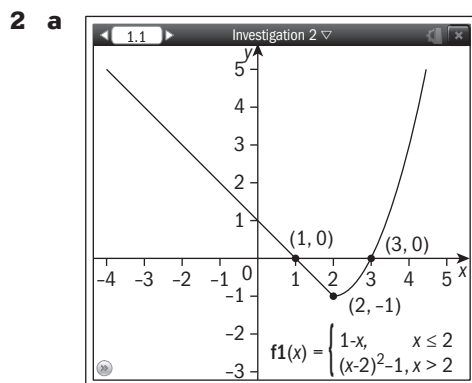
Asymptotes: $x = 1, y = 1$



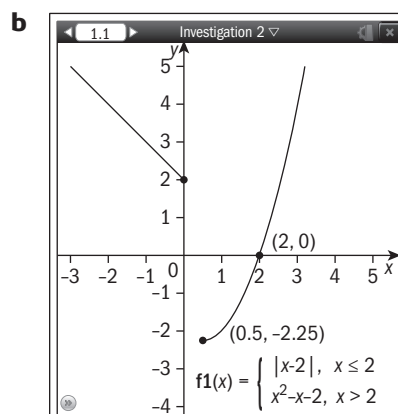
Asymptote: $y = 1$



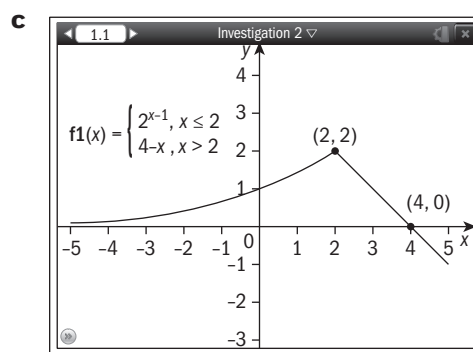
no asymptotes



no asymptotes



no asymptotes

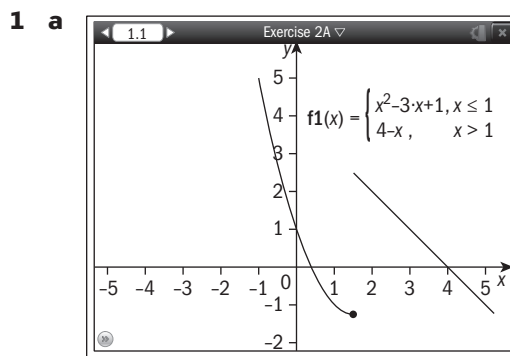


Asymptote: $y = 0$

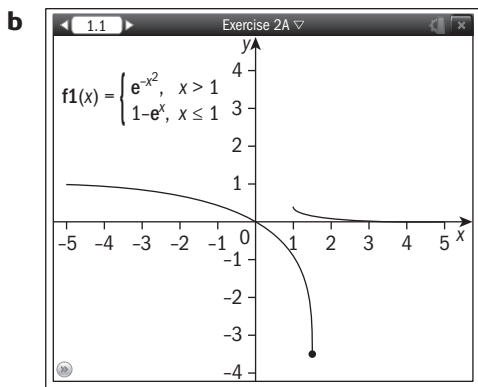
3 a $\lim_{x \rightarrow 1} ((4x-1)^2 - 5) = (4-1)^2 - 5 = 9 - 5 = 4$

b $\lim_{x \rightarrow 0} \left(\frac{\tan(2x)}{2x} - 3 \right)^3 = (1-3)^3 = (-2)^3 = -8$

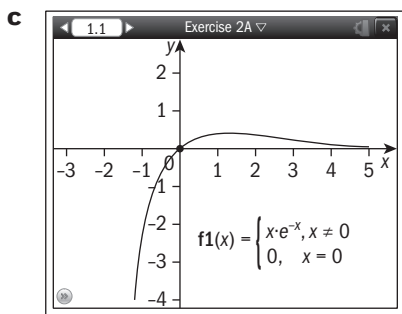
Exercise 2A



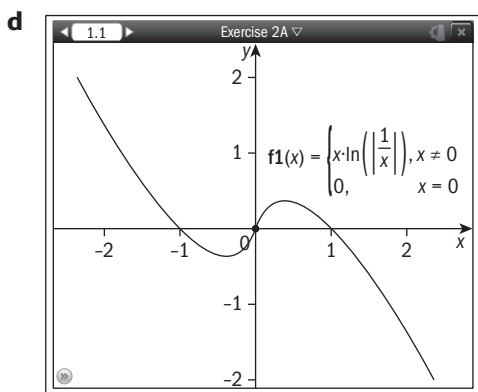
discontinuous at $x = 1$



discontinuous at $x = 1$



continuous in $\mathbb{R}$



continuous in $\mathbb{R}$

- 2** Consider the sequences $u_n = \frac{1}{2\pi n}$ and $v_n = \frac{1}{\frac{\pi}{2} + \pi n}$

$$\lim_{x \rightarrow \infty} g(u_n) = 1 \neq \lim_{x \rightarrow \infty} g(v_n) = 0.$$

$$\lim_{x \rightarrow \infty} g(x) \text{ does not exist}$$

- 3** Apply Squeeze theorem:

$$\lim_{x \rightarrow 0} \left(x \sin\left(\frac{1}{x}\right) \right) = 0 = f(0) \text{ as}$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \text{ for all } x \in \mathbb{R}.$$

4 a $\lim_{x \rightarrow a} (f + g)(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
 $= f(a) + g(a) = (f + g)(a).$

b $\lim_{x \rightarrow a} (f - g)(x) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
 $= f(a) - g(a) = (f - g)(a).$

c $\lim_{x \rightarrow a} (f \cdot g)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
 $= f(a) \cdot g(a) = (f \cdot g)(a).$

d $\lim_{x \rightarrow a} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g} \right)(a)$
 for $g(a) \neq 0.$

Investigation -Composition of continuous functions

a $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x + 1) = 1 \neq f(0) = 0$

and $\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x) = 1 \neq g(1) = 0$

but $\lim_{x \rightarrow 0} f(g(x)) = \lim_{x \rightarrow 0} f(0) = 0 = f(g(0))$

and $\lim_{x \rightarrow 1} g(f(x)) = \lim_{x \rightarrow 1} g(2) = 2 = g(f(1))$

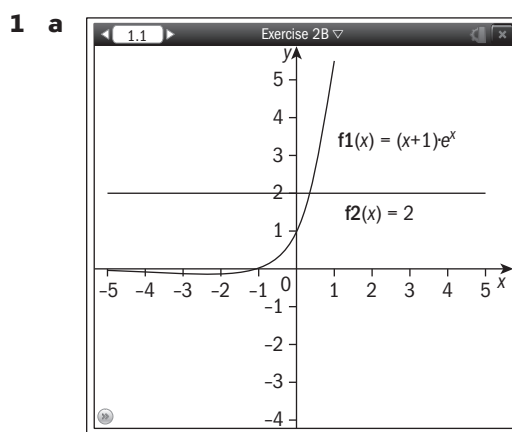
b $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x + 1) = 2 = f(1)$

and $\lim_{x \rightarrow 1} h(x) = \lim_{x \rightarrow 1} (x - 1) = 0 = h(1)$

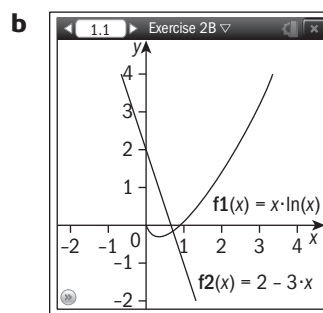
but $\lim_{x \rightarrow 1^+} h(f(x)) = \lim_{x \rightarrow 1^+} h(x + 1) = 3 \neq \lim_{x \rightarrow 1^-} h(f(x))$
 $= \lim_{x \rightarrow 1^-} h(x + 1) = 1$

- c** g is continuous at $x = a$ and f is defined and continuous at $x = g(a).$

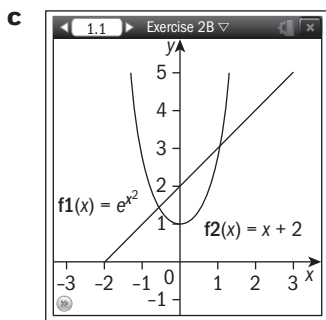
Exercise 2B



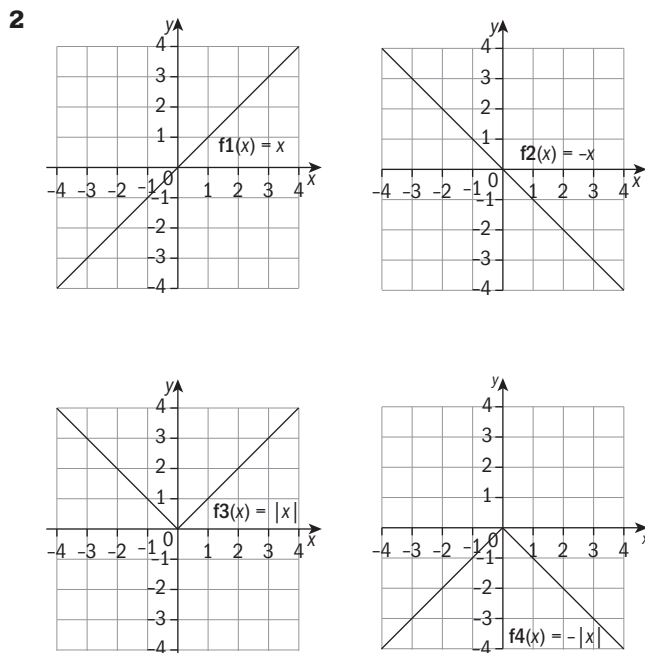
Apply theorem 5 (Bolzano's) to a closed interval that contains the point of intersection of the graphs (e.g. $[0, 1]$).



Apply theorem 5 (Bolzano's) to a closed interval that contains the point of intersection of the graphs (e.g. $[0, 1]$).



Apply theorem 5 (Bolzano's) to closed intervals that contains the point of intersection of the graphs (e.g. $[-1, 0]$ and $[0, 2]$).



Reasons: $(f(x))^2 = x^2 \Rightarrow f(x) = \pm x$ and, as f is continuous, the function can just change from $f(x) = x$ to $f(x) = -x$ at $x = 0$, so we also have $f(x) = |x|$ and $f(x) = -|x|$.

- 3 Suppose that $f(x_1) \neq f(x_2)$, let's say $f(x_1) < f(x_2)$ for $x_1, x_2 \in [a, b]$. Let $y \in]f(x_1), f(x_2)[\cap (\mathbb{R} \setminus \mathbb{Q})$. By Bolzano's theorem, there must exist at least a value $x \in]x_1, x_2[$ such that $f(x) = y$. But $y \notin \mathbb{Q}$ which contradicts the condition that $f(x) \in \mathbb{Q}$ for all values $x \in [a, b]$.
 c can be any rational number.

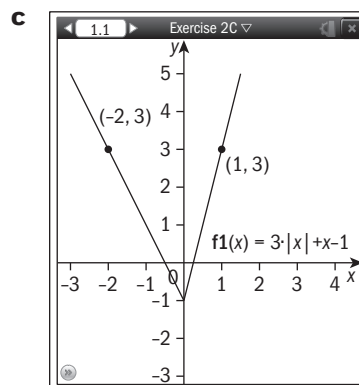
- 4 Let $p(x)$ be a polynomial with odd degree. Then $\lim_{x \rightarrow -\infty} p(x) = -\lim_{x \rightarrow +\infty} p(x)$ and by Bolzano's theorem the polynomial must have at least a zero in $]-\infty, +\infty[$.

Exercise 2C

- 1 a $f(-2) = 3|-2| - 2 - 1 = 3$ and $f(1) = 3|1| + 1 - 1 = 3$

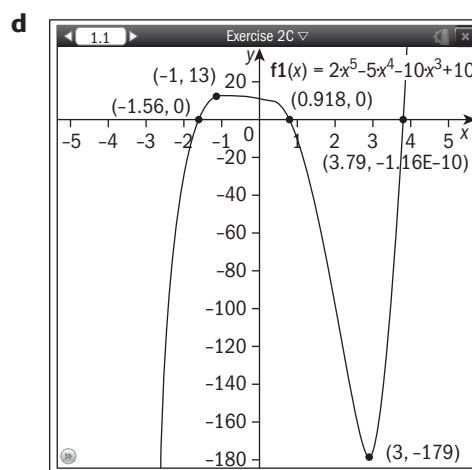
b $f'(x) = \begin{cases} 4, & x > 0 \\ -2, & x < 0 \end{cases}$

This function clearly has no zeros in any interval.



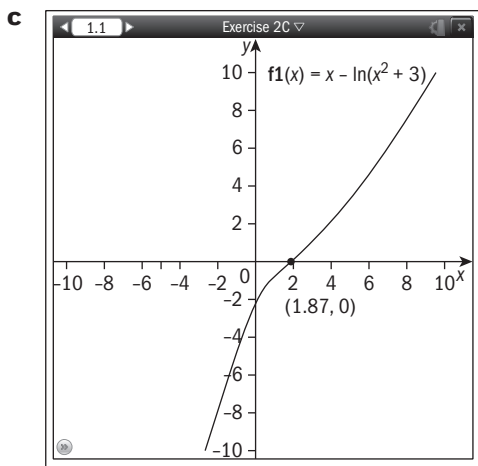
This function is not differentiable at $x = 0$, so we cannot use Rolle's Theorem.

- 2 a $f(-1) = 13$ and $f(1) = -3$. By Bolzano's theorem, as the function changes sign on the interval $[-1, 1]$, it has at least a zero on $] -1, 1[$.
- b $f'(x) = 10x^4 - 20x^3 - 30x^2$
 $f'(x) = 0 \Rightarrow 10x^2(x^2 - 2x - 3) = 0$
 $\Rightarrow x = 0, x = -1, x = 3$
- c If f had another zero on $] -1, 1[$, then its derivative would have also a zero on this interval which contradict the result in part b.



- 3 a $f'(x) = 1 - \frac{2x}{x^2 + 3} = \frac{x^2 - 2x + 3}{x^2 + 3} = \frac{(x-1)^2 + 2}{x^2 + 3} > 0$ for all $x \in \mathbb{R}$.

- b If the equation $f(x) = 0$ had more than a real solution, let's say, $a < b$ were both real solutions of $f(x) = 0$, then by Rolle's theorem $f'(x) = 0$ would have at least a solution on the interval $]a, b[$ which contradicts the result proved in part a.



No maximum nor minimum points

- 4** Let $f(x) = x^3 + 3x - 2$. This function is continuous and differentiable on any interval of real numbers. As $f(0) = -2 < 0$ and $f(1) = 2 > 0$ by Bolzano's theorem the function must have at least a zero on $]0, 1[$. As $f'(x) = 3x^2 + 3 > 0$ for all $x \in \mathbb{R}$ and therefore f' has no real zeros, the function f cannot have more than one real zero.

- 5 a** Let $f(x) = \sin(x)$ for $0 < x < \frac{\pi}{2}$. Then

$$f'(x) = \cos(x), \text{ for } 0 < x < \frac{\pi}{2}.$$

f is continuous and differentiable for $0 < x < \frac{\pi}{2}$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{\sin(x)}{x}.$$

- b** As $0 < c < x < \frac{\pi}{2}$ and $f'(c) = \frac{\sin(x)}{x}$

$$\Rightarrow \cos(c) = \frac{\sin(x)}{x} \text{ it follows that}$$

$$\cos(x) < \cos(c) < \cos(0) \Rightarrow \cos(x) < \frac{\sin(x)}{x} < 1$$

$$\text{for } 0 < x < \frac{\pi}{2}.$$

- 6 a** Let $f(x) = e^x$ for $x > 0$. Then $f'(x) = e^x$, for $x > 0$.

f is continuous and differentiable for $x > 0$.

Therefore by the MVT, there is at least one value of $c \in]0, +\infty[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{e^x - 1}{x - 0}.$$

As $0 < c < x$ and $f'(c) = \frac{e^x - 1}{x - 0} \Rightarrow \frac{e^x - 1}{x} = e^c > 1$ it follows that

$$\frac{e^x - 1}{x} > 1 \Rightarrow e^x - 1 > x \Rightarrow e^x > x + 1 \text{ for } x > 0.$$

- b** Let $f(x) = \ln(x)$ for $x > 0$. Then $f'(x) = \frac{1}{x}$, for $x > 0$.

f is continuous and differentiable for $x > 0$.

Therefore by the MVT, there is at least one value of $c \in]x, x + 1[$ such that

$$\begin{aligned} f'(c) &= \frac{f(x+1) - f(x)}{(x+1) - x} \Rightarrow f'(c) \\ &= \frac{\ln(x+1) - \ln(x)}{1} \Rightarrow f'(c) = \ln\left(\frac{x+1}{x}\right). \end{aligned}$$

As $x + 1 > c > x > 0 \Rightarrow \frac{1}{x+1} < \frac{1}{c} < \frac{1}{x}$ and

$$f'(c) = \frac{1}{c} = \ln\left(\frac{x+1}{x}\right) \text{ it follows that}$$

$$\ln\left(\frac{x+1}{x}\right) < \frac{1}{x} \text{ for } x > 0.$$

- c** Let $f(x) = \tan(x)$ for $0 < x < \frac{\pi}{2}$. Then

$$f'(x) = \frac{1}{\cos^2(x)}, \text{ for } 0 < x < \frac{\pi}{2}.$$

f is continuous and differentiable for $0 < x < \frac{\pi}{2}$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) \\ &= \frac{\tan(x) - \tan(0)}{x} \Rightarrow f'(c) = \frac{\tan(x)}{x}. \end{aligned}$$

As $0 < c < x < \frac{\pi}{2} \Rightarrow \cos^2(c) < 1 \Rightarrow \frac{1}{\cos^2(c)} > 1$ and

$$f'(c) = \frac{1}{\cos^2(c)} \text{ it follows that}$$

$$\frac{\tan(x)}{x} > 1 \Rightarrow \tan(x) > x \text{ for } 0 < x < \frac{\pi}{2}.$$

- d** Let $f(x) = \arcsin(x)$ for $0 < x < 1$. Then

$$f'(x) = \frac{1}{\sqrt{1-x^2}}, \text{ for } 0 < x < 1.$$

f is continuous and differentiable for $0 < x < 1$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$\begin{aligned} f'(c) &= \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) \\ &= \frac{\arcsin(x) - \arcsin(0)}{x} \Rightarrow f'(c) = \frac{\arcsin(x)}{x}. \end{aligned}$$

As $0 < c < x < 1$ and $f'(c) = \frac{1}{\sqrt{1-c^2}} > 1$ it follows that

$$\frac{\arcsin(x)}{x} > 1 \Rightarrow \arcsin(x) > x \text{ for } 0 < x < 1.$$

- 7 We need to show that $f'(x) = g'(x)$ and it follows from corollary 2 of Mean Value Theorem that $g(x) = f(x) + c$ for some real constant c .

$$g'(x) = \frac{2}{\left(\frac{1+x}{1-x}\right)^2 + 1} = \frac{2}{\frac{(1+x)^2 + (1-x)^2}{(1-x)^2}} = \frac{2}{2+2x^2} = \frac{1}{1+x^2} = f'(x)$$

- 8 We need to show that $f'(x) > 0$ for $x \in]0, 2[$ and it follows from corollary 3 of Mean Value Theorem that f is increasing on $]0, 2[$.

$$f'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x} > 0 \text{ when } 2x - x^2 > 0, \text{ i.e., for } x \in]0, 2[.$$

- 9 Consider $h(x) = f(x) - g(x)$.

$$f'(x) = g'(x) \Rightarrow h'(x) = 0.$$

By corollary 1 of Mean Value Theorem to $h(x) = f(x) - g(x)$ and it follows that

$$h(x) = c \Rightarrow f(x) - g(x) = c \Rightarrow f(x) = g(x) + c.$$

QED

- 10 Consider any two points $a < b$ in I . Then

$$f'(x) = \frac{f(b) - f(a)}{b - a} < 0 \Rightarrow f(b) - f(a) < 0 \Rightarrow f(b) < f(a)$$

Therefore, for any two values $a < b$ which means that f is decreasing on I .

QED

Investigation

a e.g.

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	3	1.767766953	1.732050808	2.062072616
2	4	2.121320344	2	6.066017178
...	...	...	...	...
9	11	4.596194078	3.31662479	38.58046563
10	12	4.949747468	3.464101615	42.88690166

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.1	1.449568901	1.449137675	0.029757477
2	2.2	1.48492424	1.483239697	0.113571871
...	...	...	...	...
9	2.9	1.732411614	1.702938637	1.730712821
10	3	1.767766953	1.732050808	2.062072616

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.01	1.417749096	1.417744688	0.000310945
2	2.02	1.42128463	1.42126704	0.001237616
...	...	...	...	...
9	2.09	1.446033368	1.445683229	0.024219555
10	2.1	1.449568901	1.449137675	0.029757477

x1=	5			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	5.01	2.238304045	2.238302929	4.99002E-05
2	5.02	2.240540113	2.24053565	0.000199203
...	...	...	...	...
9	5.09	2.256192589	2.256102835	0.00397831
10	5.1	2.258428657	2.258317958	0.004901841

For this function the error decreases as h decreases and as x_1 increases.

b

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	3	14.7781122	20.08553692	26.42411177
2	4	22.1671683	54.59815003	59.39941503
...	...	...	...	...
9	11	73.89056099	59874.14172	99.8765902
10	12	81.27961709	162754.7914	99.95006008

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.01	7.46294666	7.463317347	0.004966791
2	2.02	7.536837221	7.538324934	0.019735323
...	...	...	...	...
9	2.09	8.054071148	8.084915164	0.381500805
10	2.1	8.127961709	8.166169913	0.467884016

x1=	5			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	5.01	149.8972907	149.9047361	0.004966791
2	5.02	151.3814223	151.4113038	0.019735323
...	...	...	...	...
9	5.09	161.7703434	162.3898621	0.381500805
10	5.1	163.254475	164.0219073	0.467884016

x1=	10			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	10.01	22246.73045	22247.83546	0.004966791
2	10.02	22466.99511	22471.42992	0.019735323
...	...	...	...	...
9	10.09	24008.84772	24100.79243	0.381500805
10	10.1	24229.11237	24343.00942	0.467884016

For this function the error decreases as h decreases but does not seem to depend on x_1 .

c

x ₁ =	2			
n	x ₂	f(x ₂) appro	f(x ₂)	Relative Error (%)
1	3	1.193147181	1.098612289	8.604936688
2	4	1.693147181	1.386294361	22.13475204
...	...	...	...	...
9	11	5.193147181	2.397895273	116.5710588
10	12	5.693147181	2.48490665	129.1090968

x ₁ =	2			
n	x ₂	f(x ₂) appro	f(x ₂)	Relative Error (%)
1	2.1	0.743147181	0.741937345	0.16306442
2	2.2	0.793147181	0.78845736	0.594809616
...	...	...	...	...
9	2.9	1.143147181	1.064710737	7.366925198
10	3	1.193147181	1.098612289	8.604936688

x ₁ =	2			
n	x ₂	f(x ₂) appro	f(x ₂)	Relative Error (%)
1	2.01	0.698147181	0.698134722	0.001784539
2	2.02	0.703147181	0.703097511	0.007064333
...	...	...	...	...
9	2.09	0.738147181	0.737164066	0.13336442
10	2.1	0.743147181	0.741937345	0.16306442

x ₁ =	5			
n	x ₂	f(x ₂) appro	f(x ₂)	Relative Error (%)
1	5.01	1.611437912	1.611435915	0.000123948
2	5.02	1.613437912	1.613429934	0.00049452
...	...	...	...	...
9	5.09	1.627437912	1.627277831	0.009837403
10	5.1	1.629437912	1.62924054	0.012114399

For this function the error decreases as h decreases and decreases as x_1 increases.

Exercise 2D

1 a $\lim_{x \rightarrow 0} \frac{x^2 - 3x}{\sin x} = \lim_{x \rightarrow 0} \frac{2x - 3}{\cos x} = -\frac{3}{1} = -3$

b $\lim_{x \rightarrow 0} \frac{e^{2x} - e^{-3x}}{x} = \lim_{x \rightarrow 0} \frac{2e^{2x} + 3e^{-3x}}{1} = 5$

c $\lim_{x \rightarrow 0} \frac{\cos(x) + 2x - 1}{3x} = \lim_{x \rightarrow 0} \frac{-\sin(x) + 2}{3} = \frac{2}{3}$

d $\lim_{x \rightarrow 0} \frac{e^{2x} - e^{-x} - 2}{1 - \cos(2x)} = \lim_{x \rightarrow 0} \frac{2e^{2x} + e^{-x}}{2\sin(2x)} = \frac{3}{0} = \infty$

e $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\ln(\cot(x))}{\ln(\cos(x))} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\frac{-\csc^2(x)}{\cot(x)}}{\frac{-\sin(x)}{\cos(x)}} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{-\csc^2(x)}{-\sin(x)} = 1$

f $\lim_{x \rightarrow 0} \frac{x(1 - e^x)}{\ln(\cos x)} = \lim_{x \rightarrow 0} \frac{1 - e^x + xe^x}{-\sin x} = \lim_{x \rightarrow 0} \frac{\cos x(1 - e^x + xe^x)}{-\sin x}$
 $= \lim_{x \rightarrow 0} \frac{-\sin x(1 - e^x + xe^x) + \cos x(-e^x + e^x + xe^x)}{-\cos x} = 0$

g $\lim_{x \rightarrow 0} \frac{x - \arcsin x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{1 - \frac{1}{\sqrt{1-x^2}}}{\frac{2 \sin x \cos x}{\sin(2x)}} = \lim_{x \rightarrow 0} \frac{2x}{2 \cos(2x)} = 0$

h $\lim_{x \rightarrow 0^+} \frac{e^{2x} - \cos(3x)}{x \sin(2x)} = \lim_{x \rightarrow 0} \frac{2e^{2x} + 3 \sin(3x)}{\sin(2x) + x \cos(2x)} = \frac{2}{0^+} = +\infty$

i $\lim_{x \rightarrow \pi} \frac{1 + \cos(x)}{x^2 \sin(x)} = \lim_{x \rightarrow \pi} \frac{-\sin(x)}{2x \sin(x) + x^2 \cos(x)} = 0$

2 a $\lim_{x \rightarrow 1} \frac{x^2 + 3x - 2}{x^2 - x + 2} = \frac{2}{2} = 1 \neq 6$

b $\lim_{x \rightarrow 1} \frac{1 - \cos(x)}{x^2 + x} = \frac{1 - \cos(1)}{2} \neq \frac{\cos(1)}{2}$

In both parts *a* and *b* there is no indeterminate form and therefore L'Hôpital Rule cannot be applied, ie, this is not a valid method to find these limits.

3 a $\lim_{x \rightarrow +\infty} \frac{e^{5x} + x}{4x + 1} = \lim_{x \rightarrow +\infty} \frac{5e^{5x} + 1}{4} = +\infty$

b $\lim_{x \rightarrow +\infty} \frac{x^2}{x(e^x + 1)} = \lim_{x \rightarrow +\infty} \frac{x}{e^x + 1} = \lim_{x \rightarrow +\infty} \frac{1}{e^x} = \frac{1}{+\infty} = 0$

c $\lim_{x \rightarrow +\infty} \frac{\ln(e^x + 1)}{x + 3} = \lim_{x \rightarrow +\infty} \frac{\frac{e^x}{e^x + 1}}{1} = \lim_{x \rightarrow +\infty} \frac{e^x}{e^x} = 1$

4 a $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$

$\therefore f$ is continuous and differentiable in any interval of its domain $\mathbb{R} \setminus \{0\}$

$g(x) = (x - 1)^3 \Rightarrow g'(x) = 3(x - 1)^2$

$\therefore g$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

b Applying Cauchy's theorem (corollary of MTV) to f and g on the interval $[1, 3]$

$-\frac{2}{3} \times 3(x - 1)^2 = -\frac{8}{x^2} \Rightarrow x^2(x - 1)^2 = 4$

$\Rightarrow x(x + 1) = \pm 2 \Rightarrow x^2 - x + 2 = 0$ or $x^2 - x - 2 = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1-8}}{2}$ or $x = \frac{1 \pm \sqrt{1+8}}{2} \Rightarrow x = 2$ or $x = -1$
no real solutions

Therefore there is exactly one solution on the interval $]1, 3[$

5 a $f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3$

$\therefore f$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

$g(x) = (x - 1)^2 \Rightarrow g'(x) = 2(x - 1) = 2x - 2$

$\therefore g$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

b Applying Cauchy's theorem (corollary of MTV) to f and g on the interval $[-4, 2]$ we would conclude that the equation had at least one solution in the interior of the interval.

$$\text{c } \frac{2(x-1)}{3x^2-3} = \frac{-24}{64} \Rightarrow 16(x-1) = -9(x^2-3)$$

$\Rightarrow 9x^2 - 16x - 43 = 0$ which has no real solution.

d Cauchy's theorem cannot be applied in part b because f' has zeros on the interval given.

Exercise 2E

1 a Indeterminate form: $\infty \times 0$;

$$\lim_{x \rightarrow 0^+} \frac{3}{x} e^{\frac{2}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{3}{x}}{e^{\frac{2}{x}}} = \lim_{x \rightarrow 0^+} \frac{-\frac{3}{x^2} e}{-\frac{2}{x^2} e^{\frac{2}{x}}} = 0$$

b Indeterminate form: $\infty \times 0$;

$$\begin{aligned} \lim_{x \rightarrow 1^+} \arctan(x-1) e^{\frac{3}{x-1}} &= \lim_{x \rightarrow 1^+} \frac{\arctan(x-1)}{e^{-\frac{3}{x-1}}} \\ &= \lim_{x \rightarrow 1^+} \frac{\frac{1}{(x-1)^2+1}}{\frac{-\frac{3}{x-1}}{(x-1)^2}} = \lim_{x \rightarrow 1^+} \frac{1}{(x-1)^2+1} \cdot \lim_{x \rightarrow 1^+} \frac{e^{\frac{3}{x-1}}}{3} \\ &= \lim_{x \rightarrow 1^+} \frac{-\frac{3}{(x-1)^2} e^{\frac{3}{x-1}}}{-\frac{6}{(x-1)^3}} = \lim_{x \rightarrow 1^+} \frac{e^{\frac{3}{x-1}}}{2} = \lim_{x \rightarrow 1^+} \frac{-\frac{9}{(x-1)^2} e^{\frac{3}{x-1}}}{-\frac{6}{(x-1)^2}} = \infty \end{aligned}$$

c Indeterminate form: $\infty \times 0$

$$\begin{aligned} \lim_{x \rightarrow 0^+} (\sin(x))(\ln(x) - \cot(x)) &= \lim_{x \rightarrow 0^+} \frac{\ln(x) - \cot(x)}{\frac{1}{\sin(x)}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x} + \frac{1}{\sin^2(x)}}{\frac{-\cos(x)}{\sin^2(x)}} = \lim_{x \rightarrow 0^+} \frac{\frac{\sin^2(x) - x}{x \sin^2(x)}}{\frac{-\cos(x)}{\sin^2(x)}} \\ &= \lim_{x \rightarrow 0^+} \frac{x - \sin^2(x)}{x \cos(x)} = \lim_{x \rightarrow 0^+} \frac{1 - 2\sin(x)\cos(x)}{\cos(x) - x \sin(x)} = 1 \end{aligned}$$

d Indeterminate form: 1^∞

$$\begin{aligned} \lim_{x \rightarrow 0^+} (x + e^x)^{\frac{2}{x}} &= e^{\lim_{x \rightarrow 0^+} \frac{\ln(x + e^x)}{x}} \text{ and } \\ \lim_{x \rightarrow 0^+} \ln(x + e^x)^{\frac{2}{x}} &= 2 \lim_{x \rightarrow 0^+} \frac{\ln(x + e^x)}{x} \\ &= 2 \lim_{x \rightarrow 0^+} \frac{1 + e^x}{x + e^x} = 4 \\ \therefore \lim_{x \rightarrow 0^+} (x + e^x)^{\frac{2}{x}} &= e^4 \end{aligned}$$

e Indeterminate form: 0^0

$$\begin{aligned} \lim_{x \rightarrow 0^+} (\sin(2x))^{3x} &= e^{\lim_{x \rightarrow 0^+} \ln(\sin(2x))^{3x}} \text{ and } \\ \lim_{x \rightarrow 0^+} \ln(\sin(2x))^{3x} &= \lim_{x \rightarrow 0^+} \frac{\ln(\sin(2x))}{\frac{1}{3x}} = \lim_{x \rightarrow 0^+} \frac{\frac{2 \cos(2x)}{\sin(2x)}}{-\frac{1}{3x^2}} \\ &= \lim_{x \rightarrow 0^+} \left(-\frac{6x^2 \cos(2x)}{\sin(2x)} \right) \\ &= \lim_{x \rightarrow 0^+} \left(-\frac{12x \cos(2x) + 12x^2 \sin(2x)}{2 \cos(2x)} \right) = 0 \\ \therefore \lim_{x \rightarrow 0^+} (\sin(2x))^{3x} &= e^0 = 1 \end{aligned}$$

f Indeterminate form: 1^∞

$$\begin{aligned} \lim_{x \rightarrow 0^+} (1+x)^{\frac{5}{x}} &= e^{\lim_{x \rightarrow 0^+} \ln(1+x)^{\frac{5}{x}}} \text{ and } \lim_{x \rightarrow 0^+} \ln(1+x)^{\frac{5}{x}} \\ &= 5 \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} = 5 \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = 5 \\ \therefore \lim_{x \rightarrow 0^+} (1+x)^{\frac{5}{x}} &= e^5 \end{aligned}$$

g Indeterminate form: $\infty \times 0$

$$\lim_{x \rightarrow 0} (2x \cot(5x)) = \lim_{x \rightarrow 0} \frac{2x}{\tan(5x)} = \lim_{x \rightarrow 0^+} \frac{2}{5} = \frac{2}{5}$$

h Indeterminate form: 1^∞

$$\begin{aligned} \lim_{x \rightarrow 0} \ln(e-x)^{\cot(x)} &= e^{\lim_{x \rightarrow 0} \cot(x) \ln(e-x)} \text{ and } \\ \lim_{x \rightarrow 0} \cot(x) \ln(e-x) &= \lim_{x \rightarrow 0} \frac{\ln(e-x)}{\tan(x)} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{1}{e-x}}{\frac{1}{\cos^2(x)}} = \lim_{x \rightarrow 0} \left(-\frac{\cos^2(x)}{e-x} \right) = -\frac{1}{e} \\ \therefore \lim_{x \rightarrow 0} \ln(e-x)^{\cot(x)} &= e^{-\frac{1}{e}} \end{aligned}$$

i Indeterminate form: 1^∞

$$\lim_{x \rightarrow 0} (e^{e^x-1})^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{\ln(e^{e^x-1})}{x}} = e^{\lim_{x \rightarrow 0} \frac{e^x-1}{x}} = e^{\lim_{x \rightarrow 0} \frac{e^x}{1}} = e$$

k Indeterminate form: $\infty - \infty$

$$\begin{aligned} \lim_{x \rightarrow 0^+} (\ln(x) - \ln(\sin(x))) &= \ln \left(\lim_{x \rightarrow 0^+} \left(\frac{x}{\sin x} \right) \right) \\ &= \ln \left(\lim_{x \rightarrow 0^+} \left(\frac{1}{\cos x} \right) \right) = \ln 1 = 0 \end{aligned}$$

l Indeterminate form: $\infty \times 0$

$$\lim_{x \rightarrow -\infty} (x e^x) = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = 0$$

m Indeterminate form: $\infty - \infty$

$$\begin{aligned}\lim_{x \rightarrow 0^+} (\ln(e^x + 1) - \ln(e^x - 1)) &= \ln \left(\lim_{x \rightarrow 0^+} \frac{e^x + 1}{e^x - 1} \right) \\ &= \ln \left(\lim_{x \rightarrow 0^+} \left(\frac{e^x}{e^x} \right) \right) = \ln 1 = 0\end{aligned}$$

n Indeterminate form: $\infty \times 0$

$$\begin{aligned}\lim_{x \rightarrow +\infty} \left(x \sin \left(\frac{1}{x} \right) \right) &= \lim_{x \rightarrow +\infty} \frac{\sin \left(\frac{1}{x} \right)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x^2} \cos \left(\frac{1}{x} \right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow +\infty} \cos \left(\frac{1}{x} \right) = 1\end{aligned}$$

o Indeterminate form: ∞^0

$$\begin{aligned}\lim_{x \rightarrow +\infty} (1+x)^{\frac{1}{x}} &= e^{\lim_{x \rightarrow +\infty} \frac{\ln(1+x)}{x}} \text{ and} \\ \lim_{x \rightarrow +\infty} \ln(1+x)^{\frac{1}{x}} &= \lim_{x \rightarrow +\infty} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow +\infty} \frac{1}{1+x} = 0 \\ \therefore \lim_{x \rightarrow +\infty} (1+x)^{\frac{1}{x}} &= e^0 = 1\end{aligned}$$

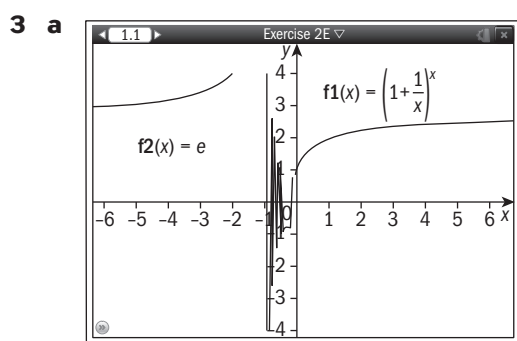
2 a $a = \lim_{x \rightarrow 0} \left(\frac{\sin(3x)}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{3 \cos(3x)}{1} \right) = 3$

b $\lim_{x \rightarrow 0^-} (b-x) = \lim_{x \rightarrow 0^+} (e^x + 4x)^{\frac{1}{x}}$

$$\Rightarrow b = e^{\lim_{x \rightarrow 0^+} \frac{\ln(e^x + 4x)}{x}} = e^{\lim_{x \rightarrow 0^+} \frac{e^x + 4}{1}} = e^5$$

c $c = \lim_{x \rightarrow 0} \left(\frac{1 - \cos(x)}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{3 \sin(x)}{2x} \right)$

$$= \lim_{x \rightarrow 0} \left(\frac{-3 \cos(x)}{2} \right) = -\frac{3}{2}$$



$\therefore \lim_{x \rightarrow +\infty} f(x) = e$

b $g(h) = \frac{\ln(1+h)}{h}$

c $\lim_{h \rightarrow 0^+} \frac{\ln(1+h)}{h} = \lim_{h \rightarrow 0^+} \frac{1}{1+h} = 1.$

Therefore $\lim_{x \rightarrow +\infty} f(x) = e^{\lim_{h \rightarrow 0^+} g(h)} = e^1 = e$

Exercise 2F

1 $\lim_{x \rightarrow 2^+} (ax^2 - x) = \lim_{x \rightarrow 2} (x^3 + bx)$

$$\Rightarrow 4a - 2 = 8 + 2b \Rightarrow 2a - b = 5$$

and $\lim_{x \rightarrow 2^-} (2ax - 1) = \lim_{x \rightarrow 2} (3x^2 + b)$

$$\Rightarrow 4a - 1 = 12 + b \Rightarrow 4a - b = 13$$

Solving simultaneously we obtain $a = 4, b = 3$

2 $\lim_{x \rightarrow 0^+} \frac{e^{ax} - 1}{x} = \lim_{x \rightarrow 0^-} (bx + 3) \Rightarrow \lim_{x \rightarrow 0^+} \frac{ae^{ax}}{1} = 3 \Rightarrow a = 3$

and $\lim_{x \rightarrow 0^+} \frac{axe^{ax} - (e^{ax} - 1)}{x^2} = \lim_{x \rightarrow 0^-} b$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{ae^{ax} + a^2 xe^{ax} - ae^{ax}}{2x} = b \Rightarrow \frac{a^2}{2} = b$$

Solving simultaneously we obtain $a = 3, b = \frac{9}{2}$

3 $\lim_{x \rightarrow 1^+} (\ln(ax)) = \lim_{x \rightarrow 1^-} (\arctan(bx)) \Rightarrow \ln(a) = \arctan(b)$

$$\lim_{x \rightarrow 1^+} \frac{1}{x} = \lim_{x \rightarrow 1^-} \frac{b}{1 + b^2 x^2} \Rightarrow 1 = \frac{b}{1 + b^2} \Rightarrow 1 + b^2 = b$$

which has no real solution

Exercise 2G

1 As $\lim_{x \rightarrow +\infty} \frac{x^2 + 2x + 1}{e^{3x} + x} = \lim_{x \rightarrow +\infty} \frac{2x + 2}{3e^{3x} + 1} = \lim_{x \rightarrow +\infty} \frac{2}{9e^{3x}} = 0$ then

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 2n + 1}{e^{3n} + n} = 0$$

2 As $\lim_{x \rightarrow +\infty} \frac{\arctan \left(\frac{1}{x} \right)}{\ln \left(1 + \frac{2}{x} \right)} = \lim_{x \rightarrow +\infty} \frac{\frac{-1}{x^2}}{\frac{2}{x^2}} = \lim_{x \rightarrow +\infty} \frac{1 + \frac{2}{x}}{1 + \frac{1}{x^2}} = 1$

then $\lim_{n \rightarrow +\infty} \frac{\arctan \left(\frac{1}{n} \right)}{\ln \left(1 + \frac{2}{n} \right)} = 1$

3 As $\lim_{x \rightarrow +\infty} \left(\frac{x}{2^x} \right)^{\frac{1}{x}} = e^{\lim_{x \rightarrow +\infty} \frac{\ln \left(\frac{x}{2^x} \right)}{x}} = e^{\lim_{x \rightarrow +\infty} \frac{\ln x - x \ln 2}{x}} = e^{\lim_{x \rightarrow +\infty} \frac{1 - \ln 2}{1}}$

$$= e^{-\ln 2} = \frac{1}{2} \text{ then } \lim_{n \rightarrow +\infty} \left(\frac{n}{2^n} \right)^{\frac{1}{n}} = \frac{1}{2}$$

Review exercise

- 1 a** Consider the function defined by $f(x) = 2 \arctan(x) - x$. The solutions of the equation $2 \arctan(x) = x$ are the zeros of f . Suppose that $a < b$ are two consecutive zeros of f . Then by Rolle's theorem f' must have at least a zero

on the interval $]a, b[$ (as f clearly satisfies the condition of being continuous and differentiable on any interval of real numbers).

$$\text{As } f'(x) = \frac{2}{1+x^2} - 1 = 0 \Rightarrow 1+x^2 = 2 \Rightarrow x = \pm 1$$

f can have at most a zero on

$]-\infty, -1[$, a zero on $]-1, 1[$ and a zero on $]1, +\infty[$.

b $f(0) = 0$

$\therefore x = 0$ is solution of $2 \arctan x = x$

Therefore the equation has exactly one solution on $]-1, 1[$.

2 a Let $f(x) = \cos(x) - 2x$. Then $f'(x) = \sin(x) - 2$.

$f'(x) = 0 \Rightarrow \sin(x) = -2$ which is impossible.

(1) As f is continuous and differentiable on any interval of real numbers and f' has no zeros, by Rolle's theorem, f has at most one real zero.

(2) As $f(0) = \cos(0) - 2 \times 0 = 1 > 0$ and

$$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - 2 \times \frac{\pi}{4} = \frac{\sqrt{2}-\pi}{2} < 0, \text{ by}$$

Bolzano's theorem, f has at least a zero on

$$\left]0, \frac{\pi}{4}\right[.$$

By (1) and (2) we can conclude that f has exactly a zero on $\left]0, \frac{\pi}{4}\right[$.

b $\lim_{x \rightarrow 0} (\cos(x) - 2x)^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{\ln(\cos(x) - 2x)}{x}}$

$$= e^{\lim_{x \rightarrow 0} \frac{-\sin(x) - 2}{\cos(x) - 2x}} = e^{-2}$$

c $\lim_{x \rightarrow +\infty} (\cos(x) - 2x) = -\infty$

3 a Consider the function defined by $f(t) = \sin(t)$. f is continuous and differentiable in any interval $[x, y]$, $x < y$.

By MTV, applied to any interval $[x, y]$, $x < y$

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ where } c \in]x, y[$$

$$\text{Therefore } \left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)|$$

$$\Rightarrow |f(x) - f(y)| = |f'(c)| \cdot |x - y|, x < y$$

As $f'(t) = \cos(t)$ and $|\cos(t)| \leq 1$ for any real

value of t , $|f(x) - f(y)| = |f'(c)| \cdot |x - y|$

$$\Rightarrow |f(x) - f(y)| = |x - y|. \quad \text{QED}$$

b Consider the function defined by $f(t) = \sqrt{t}$. f is continuous and differentiable in any interval $[x, y]$, $0 < x < y$.

By MTV, applied to any interval $[x, y]$, $0 < x < y$

$$\frac{f(x) - f(y)}{x - y} = f'(c) \text{ where } c \in]x, y[$$

$$\text{Therefore } \frac{f(y) - f(x)}{y - x} = f'(c)$$

$$\Rightarrow f(y) - f(x) = f'(c) \cdot (y - x), 0 < x < y$$

$$\text{As } f'(t) = \frac{1}{2\sqrt{t}} \text{ and } \frac{1}{2\sqrt{y}} \leq \frac{1}{2\sqrt{t}} \leq \frac{1}{2\sqrt{x}} \text{ for any}$$

real value of t , $0 < x \leq t \leq y$

$$\sqrt{y} - \sqrt{x} = \frac{1}{2\sqrt{c}} \cdot (y - x)$$

$$\Rightarrow -\frac{1}{2\sqrt{y}} \cdot (x - y) \leq \sqrt{y} - \sqrt{x} \leq -\frac{1}{2\sqrt{x}} \cdot (x - y)$$

$$\Rightarrow \sqrt{x} - \frac{x - y}{2\sqrt{y}} \leq \sqrt{y} \leq \sqrt{x} - \frac{x - y}{2\sqrt{x}}. \quad \text{QED}$$

c Consider the function defined by $f(t) = \ln(x)$. f is continuous and differentiable in any interval $[x, +\infty]$, $x > 0$.

By MTV, applied to any interval $[x-1, x]$, $x > 1$

$$\frac{f(x) - f(x-1)}{x - (x-1)} = f'(c) \text{ where } c \in]x-1, x[$$

$$\text{Therefore } \frac{f(x) - f(x-1)}{x - (x-1)} = f'(c)$$

$$\Rightarrow f(x) - f(x-1) = f'(c), 0 < x-1 < c < x$$

As $f'(t) = \frac{1}{t}$ and $\frac{1}{x} < \frac{1}{t} < \frac{1}{x-1}$ for any real value of t , $0 < x-1 < t < x$

$$\ln(x) - \ln(x-1) = \frac{1}{t} \Rightarrow \ln\left(\frac{x}{x-1}\right) < \frac{1}{x-1}, x > 1. \quad \text{QED}$$

d Consider the function defined by $f(t) = \sin(x)$. f is continuous and differentiable in any interval of real numbers.

By MTV, applied to any interval $[0, x] \subset \left[0, \frac{\pi}{2}\right]$.

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \text{ where } c \in]0, x[\subset \left[0, \frac{\pi}{2}\right]$$

$$\text{Therefore } \frac{f(x) - f(0)}{x - 0} = f'(c)$$

$$\Rightarrow f(x) - f(0) = f'(c) \cdot x, 0 < c < x \leq \frac{\pi}{2}$$

As $f'(t) = \cos(t)$ and $0 < \cos(t) < 1$ for

$$0 < t < x \leq \frac{\pi}{2}$$

$$\sin(x) - \sin(0) = \cos(c) \cdot x \Rightarrow \sin(x) < x,$$

$$0 < x \leq \frac{\pi}{2}.$$

As $\sin(0) = 0$, we have $\sin(x) \leq x$, for

$$0 \leq x \leq \frac{\pi}{2}. \quad \text{QED}$$

$$4 \quad \lim_{x \rightarrow 1} \frac{x^2 - x}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x \rightarrow 1} \frac{2x - 1}{-\frac{\pi}{2} \csc^2\left(\frac{\pi x}{2}\right)} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$$

$$5 \quad \text{Let } f(x) = \cot\left(\frac{\pi x}{2}\right), x \in]0, 1[.$$

$$\begin{aligned} \text{a} \quad \lim_{x \rightarrow 0} (x \cdot f(x)) &= \lim_{x \rightarrow 0} \left(\frac{x}{\tan\left(\frac{\pi x}{2}\right)} \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{\frac{\pi}{2} \sec^2\left(\frac{\pi x}{2}\right)} \right) = \frac{2}{\pi} \end{aligned}$$

$$\begin{aligned} \text{b} \quad \lim_{x \rightarrow 0} (\ln(e^x + x)^{f(x)}) &= \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{\tan\left(\frac{\pi x}{2}\right)} \\ &= \lim_{x \rightarrow 0} \frac{\frac{e^x + 1}{e^x + x}}{\frac{\pi}{2} \sec^2\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi} = \frac{4}{\pi} \end{aligned}$$

6 a i Let f be an even function defined on $[-a, a]$, $a \in \mathbb{R}^+$.

Then $f(-x) = f(x)$ for any

$x \in [-a, a]$, $a \in \mathbb{R}^+$. By definition,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-(-h)} \quad (f \text{ even}) \\ &= \lim_{h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-(-h)} \\ &= -\lim_{-h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-h} = -f'(-x) \text{ which} \end{aligned}$$

shows that f' is odd. QED

ii Let f be an odd function defined on $[-a, a]$, $a \in \mathbb{R}^+$.

Then $f(-x) = -f(x)$ for any

$x \in [-a, a]$, $a \in \mathbb{R}^+$. By definition,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-f(-x-h) + f(-x)}{-(-h)} \quad (f \text{ odd}) \\ &= \lim_{h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-h} \\ &= \lim_{-h \rightarrow 0} \frac{f(-x-h) - f(-x)}{-h} = f'(-x) \end{aligned}$$

which shows that f' is even. QED

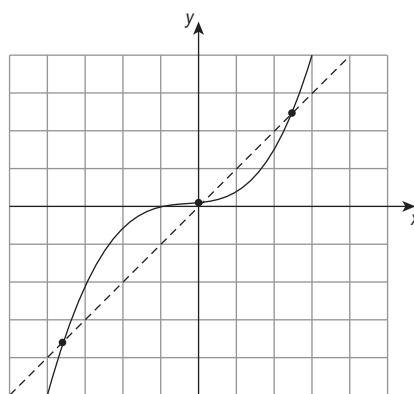
b i Let f be an even non-constant function differentiable (and therefore continuous) on $[-a, a]$, $a \in \mathbb{R}^+$. As $f(-a) = f(a)$ by the Mean value theorem, f' has at least a zero on $] -a, a[$, $a \in \mathbb{R}^+$ and, by a, as f' is odd one of these zeros is at $x = 0$ where f' changes sign and therefore f has a maximum or a minimum at $x = 0$.

ii Let f be an odd function differentiable (and therefore continuous) on $[-a, a]$, $a \in \mathbb{R}^+$. Then by a, as f' is even and $f'(-a) = f'(a)$ by the Mean value theorem, f'' has at least a zero on $] -a, a[$, $a \in \mathbb{R}^+$ and, by a, as f'' is odd one of these zeros is at $x = 0$ where f'' changes sign and therefore f has an inflexion point at $x = 0$.

c i Let f be an odd function on $[-a, a]$, $a \in \mathbb{R}^+$. Then the graph of f intersects the line $y = x$ at the origin as $f(0) = 0$.

ii Suppose that the graph of f intersects the line $y = x$ at $x = b \in]0, a]$, ie, for $x = b \Rightarrow y = b \Rightarrow f(b) = b$. But then $x = -b \Rightarrow f(-b) = -b \Rightarrow y = -b$ which means that the graph of f intersects the line $y = x$ at $x = -b \in [-a, 0[$. Therefore the graph of f intersects the line $y = x$ an odd number of times (at the origin and at pairs of points with coordinates (b, b) and $(-b, -b)$, $b \in]0, a]$. QED

d e.g.



e Applying the Mean value Theorem to the interval $[-b, 0]$, with $b \in]0, a]$ as in part c, we have $f'(c) = \frac{f(b) - f(0)}{b - 0} = \frac{b}{b} = 1$ for $c \in] -b, 0[$. As f is odd the same occurs on the interval $[0, b]$.

3

Modeling dynamic phenomena

Skills check

1 a $x e^y + x^2 = 1 \Rightarrow x e^y = 1 - x^2 \Rightarrow e^y$

$$= \frac{1-x^2}{x} \Rightarrow y = \ln \left(\frac{1-x^2}{x} \right)$$

b $\ln \left(\frac{x+1}{y} \right) = 2x \Rightarrow \frac{x+1}{y} = e^{2x} \Rightarrow y = \frac{x+1}{e^{2x}}$

or $y = (x+1)e^{-2x}$

c $\tan \left(\frac{x}{y} \right) = 1 \Rightarrow \frac{x}{y} = \frac{\pi}{4} \Rightarrow y = \frac{4x}{\pi}$
(assuming principal restriction)

d $\arccos(xy - 1) = 2x \Rightarrow xy - 1 = \cos(2x)$
 $\Rightarrow y = \frac{\cos 2x + 1}{x}$

2 a $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$
 $= x^2 e^x - 2(x e^x - \int e^x dx)$

i $x^2 e^x - 2x e^x + 2e^x + c$ or $(x^2 - 2x + 2)e^x + c$

ii $\int \sin(2x) \cdot e^x dx = \sin(2x) \cdot e^x - 2 \int \cos(2x) \cdot e^x dx$
 $\Rightarrow \int \sin(2x) \cdot e^x dx$
 $= \sin(2x) \cdot e^x - 2(\cos(2x) \cdot e^x - 2 \int \sin(2x) \cdot e^x dx)$

$$\Rightarrow 5 \int \sin(2x) \cdot e^x dx = (\sin(2x) - 2\cos(2x)) \cdot e^x + c$$

$$\Rightarrow \int \sin(2x) \cdot e^x dx = \frac{1}{5}(\sin(2x) - 2\cos(2x)) \cdot e^x + c$$

iii $\int \frac{e^x}{e^x - 1} dx = \int \frac{1}{u} du$ where $u = e^x - 1 \Rightarrow du = e^x dx$

As $\int \frac{1}{u} du = \ln|u| + c$ then $\int \frac{e^x}{e^x - 1} dx = \ln|e^x - 1| + c$

iv $\int \frac{\cos(x)}{\sin^2(x)} dx = \int \frac{1}{u^2} du$ where $u = \sin(x)$
 $\Rightarrow du = \cos(x) dx$

As $\int \frac{1}{u^2} du = -\frac{1}{u} + c$ then $\int \frac{\cos(x)}{\sin^2(x)} dx = -\frac{1}{\sin(x)} + c$.

b i $y' = \frac{1 \cdot (x-1) - x \cdot 1}{(x-1)^2} = \frac{-1}{(x-1)^2} = \frac{-1}{(x-1)^2}$
 $= -\frac{1}{|x-1|\sqrt{1-2x}}$

ii $\frac{\sin(2x)}{\sin^2(x)+1}$

3 a $\frac{dy}{dx}(x=1) = e^2 - \ln(1) = e^2$

b $\frac{dy}{dx}\left(x = \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Exercise 3A

1	EQUATION	LINEAR	ORDER	COEFFICIENTS
1		YES	3	Constant
2		NO	1	Constant
3		YES	2	Variable
4		NO	1	Constant
5		YES	1	Variable
6		YES	1	Variable
7		NO	1	Variable

2 a $y = e^{-x} \Rightarrow y' = -e^{-x}$

Substitute into $y' + y = 0$ to obtain $e^{-x} - e^{-x} = 0$
which is true for all $x \in \mathbb{R}$

$\therefore y = e^{-x}$ is solution of the differential equation $y' + y = 0$.

b $y = \sin(x) \Rightarrow y' = \cos(x) \Rightarrow y'' = -\sin(x)$

Substitute into $y'' + y = 0$ to obtain
 $-\sin(x) + \sin(x) = 0$ which is true for all $x \in \mathbb{R}$

$\therefore y = \sin(x)$ is solution of the differential equation $y'' + y = 0$.

c $y = x e^{-x} \Rightarrow y' = e^{-x} - x e^{-x}$

Substitute into $y' + y = e^{-x}$ to obtain $e^{-x} - e^{-x} = 0$
which is true for all $x \in \mathbb{R}$

$\therefore y = x e^{-x}$ is solution of the differential equation $y' + y = e^{-x}$.

3 a Differentiating implicitly $e^{xy} + x + y = 0$,

$$y e^{xy} + x e^{xy} \frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$$

$$\Rightarrow (1 + x e^{xy}) \frac{dy}{dx} = -(1 + y e^{xy})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1 + y e^{xy}}{1 + x e^{xy}} \text{ QED}$$

- b** Differentiating implicitly $x^2 + y^2 = r^2$, r constant
 $2x + 2y \frac{dy}{dx} = 0 \Rightarrow y \frac{dy}{dx} = -x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ QED

Investigation

- 1** For example, $y = e^x \Rightarrow y' = e^x$.
 Substitute into $y' - y = 0$ to obtain $e^x - e^x = 0$
 which is true for all $x \in \mathbb{R} \therefore y = e^x$ is solution of
 the differential equation $y' - y = 0$
 For example, $y = 2e^x \Rightarrow y' = 2e^x$.
 Substitute into $y' - y = 0$ to obtain $2e^x - 2e^x = 0$
 which is true for all $x \in \mathbb{R}$
 $\therefore y = 2e^x$ is solution of the differential equation
 $y' - y = 0$
 In general, $y = Ae^x \Rightarrow y' = Ae^x$.
 Substitute into $y' - y = 0$ to obtain $Ae^x - Ae^x = 0$
 which is true for all $x \in \mathbb{R}$
 $\therefore y = Ae^x$ is a general solution of the differential
 equation $y' - y = 0$.
 When $x = 0 \Rightarrow y = 3 \Rightarrow Ae^0 = 3 \Rightarrow A = 3$.
 $\therefore y = 3e^x$ is the particular solution (curve) that
 contains the point $(0, 3)$.
- 2** For example, $y = -\frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = y^2$.
 For example, $y = -\frac{1}{x-1} \Rightarrow \frac{dy}{dx} = \frac{1}{(x-1)^2} \Rightarrow \frac{dy}{dx} = y^2$
 In general, $y = -\frac{1}{x+k} \Rightarrow \frac{dy}{dx} = \frac{1}{(x+k)^2} \Rightarrow \frac{dy}{dx} = y^2$.
 When $x = 1 \Rightarrow y = 1 \Rightarrow 1 = -\frac{1}{1+k} \Rightarrow k = -2$.
 $\therefore y = -\frac{1}{x-2}$ is the particular solution (curve) that
 contains the point $(1, 1)$.
- 3** No solution as $(y'(x))^2 \geq 0$

Exercise 3B

- 1 a** $\frac{dx}{dt} = t + t^2 \Rightarrow x = \frac{t^2}{2} + \frac{t^3}{3} + c$
b $\frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln|x| + c$
c $\frac{dz}{dx} = x \cos(x) \Rightarrow z = x \sin(x) - \int \sin(x) dx$
 $\Rightarrow z = x \sin(x) + \cos(x) + c$
d $\frac{dw}{dx} = x e^{2x} \Rightarrow w = \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} dx$
 $\Rightarrow w = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$
- 2 a** $\frac{dy}{dx} = 3x + x^2 \Rightarrow y = \frac{3}{2} x^2 + \frac{1}{3} x^3 + c$
 $y(1) = 4 \Rightarrow \frac{3}{2} + \frac{1}{3} + c = 4 \Rightarrow c = \frac{13}{6}$
 $\therefore y = \frac{3}{2} x^2 + \frac{1}{3} x^3 + \frac{13}{6}$

b $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}} \Rightarrow y = \frac{1}{2} \int \frac{-2x}{\sqrt{4-x^2}} dx$
 $u = 4 - x^2 \Rightarrow du = -2x dx$

$$y = \frac{1}{2} \int \frac{1}{\sqrt{u}} du \Rightarrow y = \sqrt{u} + c$$

$$\therefore y = \sqrt{4-x^2} + c$$

$$y(0) = 1 \Rightarrow \sqrt{4} + c = 1 \Rightarrow c = -1$$

$$\therefore y = \sqrt{4-x^2} - 1$$

3 $\frac{dy}{dx} = \frac{1}{x+2} - \frac{1}{2} \sin(x) \Rightarrow y = \int \frac{1}{x+2} dx - \frac{1}{2} \int \sin(x) dx$

$$y = \ln|x+2| + \frac{1}{2} \cos(x) + c$$

$$y(0) = 2 \Rightarrow \ln 2 + \frac{1}{2} \cos(0) + c = 2$$

$$\Rightarrow c = \frac{3}{2} - \ln 2$$

$$\therefore y = \ln|x+2| + \frac{1}{2} \cos(x) + \frac{3}{2} - \ln 2$$

4 $y = \sin(kx) - kx \cos(kx)$

$$\Rightarrow \frac{dy}{dx} = k \cos(kx) - k \cos(kx) + k^2 x \sin(kx)$$

$$\Rightarrow \frac{dy}{dx} = k^2 x \sin(kx) \text{ QED}$$

5 $\frac{dy}{dx} = e^{-2x} - \frac{1}{x-1} \Rightarrow y = \int \left(e^{-2x} - \frac{1}{x-1} \right) dx$

$$\therefore y = -\frac{1}{2} e^{-2x} - \ln|x-1| + c$$

$$y(0) = 1 \Rightarrow -\frac{1}{2} - \ln 1 + c = 1 \Rightarrow c = \frac{3}{2}$$

$$\therefore y = -\frac{1}{2} e^{-2x} - \ln|x-1| + \frac{3}{2}$$

6 a $\frac{dV}{dt} = 8 \Rightarrow \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) = 8$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 8 \left(\Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2} \right)$$

b $\frac{dr}{dt} = \frac{2}{\pi r^2} \Rightarrow \int r^2 dr = \frac{2}{\pi} \int dt \Rightarrow \frac{1}{3} r^3 = \frac{2}{\pi} t + c$

$$r(1) = 5 \Rightarrow \frac{1}{3} \times 125 = \frac{2}{\pi} + c \Rightarrow c = \frac{125}{3} - \frac{2}{\pi}$$

$$\therefore \frac{1}{3} r^3 = \frac{2}{\pi} t + \frac{125}{3} - \frac{2}{\pi} \Rightarrow r = \sqrt[3]{\frac{6}{\pi} (t-1) + 125}$$

7 $v = \frac{1}{2+t^2} \Rightarrow \frac{ds}{dt} = \frac{1}{2+t^2} \Rightarrow s = \int \frac{1}{2+t^2} dt$

$$\therefore s = \frac{1}{\sqrt{2}} \arctan \left(\frac{t}{\sqrt{2}} \right) + c$$

Exercise 3C

$$\begin{aligned} 1 \quad a \quad \frac{dy}{dx} &= y - 3 \Rightarrow \frac{dy}{y-3} = dx \\ &\Rightarrow \int \frac{1}{y-3} dy = \int dx \Rightarrow \ln|y-3| = x + c \\ |y-3| &= e^{x+c} \Rightarrow y = Ae^x + 3 \end{aligned}$$

$$\begin{aligned} b \quad \frac{dy}{dx} &= \frac{1}{3y^2 + y - 1} \Rightarrow (3y^2 + y - 1)dy = dx \\ &\Rightarrow \int (3y^2 + y - 1)dy \\ &= \int dx \Rightarrow y^3 + \frac{y^2}{2} - y = x + c \end{aligned}$$

$$\begin{aligned} c \quad \frac{dy}{dx} &= y^2 + 4 \Rightarrow \frac{dy}{y^2 + 4} = dx \Rightarrow \int \frac{dy}{y^2 + 4} = \int dx \\ \therefore \frac{1}{2} \arctan\left(\frac{y}{2}\right) &= x + c \end{aligned}$$

$$\begin{aligned} d \quad \frac{dy}{dx} &= 2y + 1 \Rightarrow \frac{dy}{2y+1} = dx \Rightarrow \int \frac{1}{2y+1} dy = \int dx \\ &\Rightarrow \frac{1}{2} \int \frac{2}{2y+1} dy = \int dx \\ \therefore \frac{1}{2} \ln|2y+1| &= x + c \end{aligned}$$

$$\begin{aligned} e \quad \frac{dy}{dx} &= \cos^2(3y-1) \Rightarrow \frac{dy}{\cos^2(3y-1)} = dx \\ &\Rightarrow \int \frac{dy}{\cos^2(3y-1)} = \int dx \\ &\Rightarrow \frac{1}{3} \int \frac{3dy}{\cos^2(3y-1)} = \int dx \\ \therefore \frac{1}{3} \arctan(3y-1) &= x + c \end{aligned}$$

$$\begin{aligned} f \quad \frac{dy}{dx} &= \frac{1 + \sin^2 y}{\cos y} \Rightarrow \frac{\cos y}{1 + \sin^2 y} dy = dx \\ &\Rightarrow \int \frac{\cos y}{1 + \sin^2 y} dy = \int dx \\ \therefore \arctan(\sin y) &= x + c \end{aligned}$$

$$2 \quad a \quad e^{-x} \frac{dy}{dx} = \frac{3}{y} \Rightarrow \int y dy = 3 \int e^x dx \Rightarrow \frac{y^2}{2} = 3e^x + c$$

To check answers, differentiate both sides of the equation, e.g.

$$\begin{aligned} \frac{d}{dx} \left(\frac{y^2}{2} \right) &= \frac{d}{dx} (3e^x + c) \Rightarrow y \frac{dy}{dx} = 3e^x \\ &\Rightarrow e^{-x} \frac{dy}{dx} = \frac{3}{y} \end{aligned}$$

$$\begin{aligned} b \quad \frac{5}{y-1} \frac{dy}{dx} &= \frac{2}{x} \Rightarrow \frac{5}{y-1} dy = \frac{2}{x} dx \\ &\Rightarrow \int \frac{5}{y-1} dy = \int \frac{2}{x} dx \\ \therefore 5 \ln|y-1| &= 2 \ln|x| + c \Rightarrow (y-1)^5 = Ax^2 \end{aligned}$$

$$\begin{aligned} c \quad \frac{dy}{dx} &= \frac{y}{x} \Rightarrow \frac{1}{y} dy = \frac{1}{x} dx \Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x} dx \\ \therefore \ln|y| &= \ln|x| + c \Rightarrow y = Ax \end{aligned}$$

$$\begin{aligned} d \quad \frac{dy}{dx} &= -\frac{y^2+1}{x^2+1} \Rightarrow \frac{1}{y^2+1} dy = -\frac{1}{x^2+1} dx \\ &\Rightarrow \int \frac{1}{y^2+1} dy = -\int \frac{1}{x^2+1} dx \\ \therefore \arctan(y) &= -\arctan(x) + c \\ \text{or } y &= -\tan(\arctan(x) + c) \end{aligned}$$

$$\begin{aligned} e \quad e^{-x} \frac{dy}{dx} &= \frac{2x}{y} \Rightarrow \int y dy = 2 \int x e^x dx \\ &\Rightarrow \frac{y^2}{2} = 2x e^x - 2 \int e^x dx \\ \therefore \frac{y^2}{2} &= 2x e^x - 2e^x + c \Rightarrow y^2 = 4(x-1)e^x + c \end{aligned}$$

$$\begin{aligned} f \quad \frac{x}{y^2+1} dy &= \frac{2}{x} dx \Rightarrow \int \frac{1}{y^2+1} dy = -2 \int \frac{1}{x^2} dx \\ \therefore \arctan(y) &= -\frac{2}{x} + c \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad (1 + \cos(2x)) \frac{dy}{dx} &= 2 \sin(2x)y \Rightarrow \frac{1}{y} dy \\ &= \frac{2 \sin(2x)}{1 + \cos(2x)} dx \\ \int \frac{dy}{y} &= \int \frac{2 \sin 2x}{1 + \cos 2x} dx \\ \Rightarrow \ln|y| &= -\ln|1 + \cos 2x| + c \\ \Rightarrow y &= \frac{c}{1 + \cos 2x} \\ y(0) = 1 &\Rightarrow c = 2 \\ y &= \frac{2}{1 + \cos 2x} \Rightarrow y = \frac{2}{1 + (2 \cos^2 x - 1)} \\ \therefore y &= \sec^2 x \end{aligned}$$

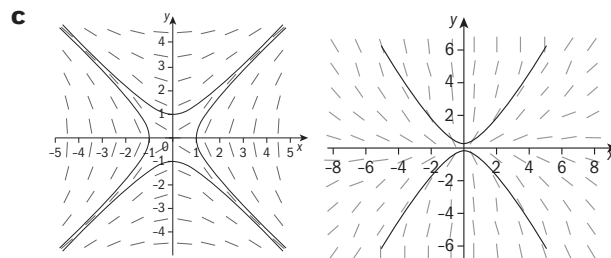
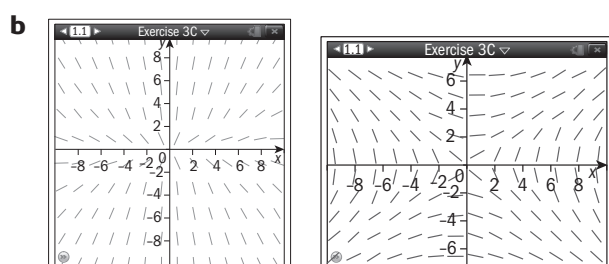
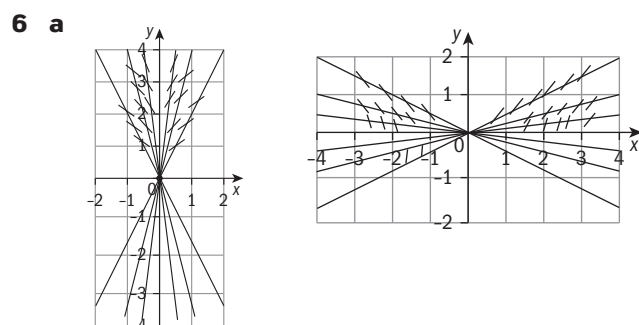
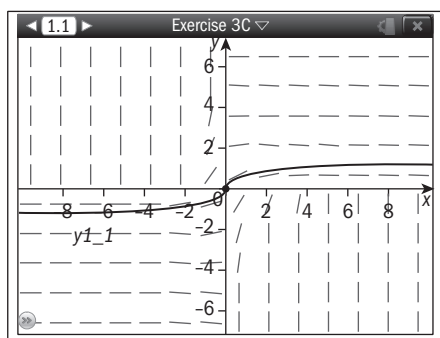
$$\begin{aligned} b \quad (1+x^2) \frac{dy}{dx} &= 1+y^2 \Rightarrow \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx \\ &\Rightarrow \int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx \\ \Rightarrow \arctan(y) &= \arctan(x) + c \\ y(0) = 2 &\Rightarrow \arctan(2) = \arctan(0) + c \\ \therefore c &= \arctan(2) \\ \therefore y &= \tan(\arctan(x) + \arctan(2)) \\ \therefore y &= \frac{x+2}{1-2x} \end{aligned}$$

c $e^x \frac{dy}{dx} + 1 = y \Rightarrow \frac{1}{y-1} dy = e^{-x} dx$
 $\Rightarrow \int \frac{1}{y-1} dy = \int e^{-x} dx$
 $\therefore \ln|y-1| = -e^{-x} + c$
 $y(0) = 2 \Rightarrow \ln 1 = -e^0 + c$
 $\therefore c = 1$
 $\therefore \ln|y-1| = 1 - e^{-x}$

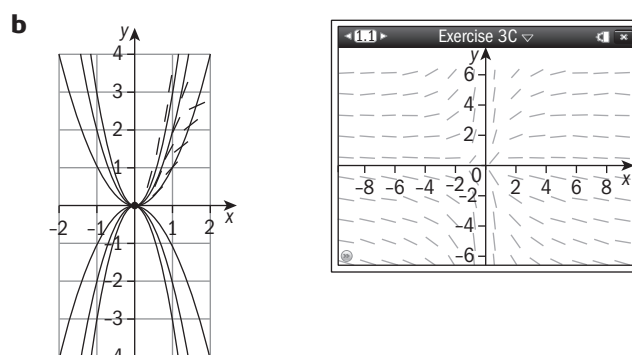
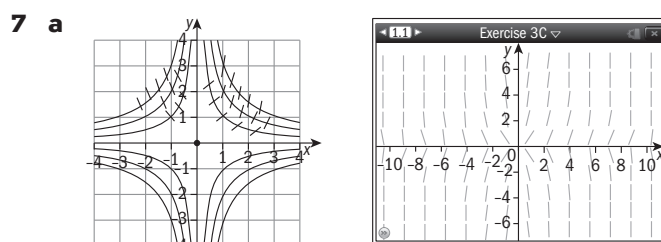
4 a $\frac{dV}{dt} = 8 \Rightarrow \frac{d}{dt} \left(\frac{4}{3} \pi r^3 \right) = 8$
 $\Rightarrow 4\pi r^2 \frac{dr}{dt} = 8 \left(\Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2} \right)$
b $\frac{dr}{dt} = \frac{2}{\pi r^2} \Rightarrow \int r^2 dr = \frac{2}{\pi} \int dt \Rightarrow \frac{1}{3} r^3 = \frac{2}{\pi} t + c$
 $r(1) = 5 \Rightarrow \frac{1}{3} \times 125 = \frac{2}{\pi} + c \Rightarrow c = \frac{125}{3} - \frac{2}{\pi}$
 $\therefore \frac{1}{3} r^3 = \frac{2}{\pi} t + \frac{125}{3} - \frac{2}{\pi} \Rightarrow r = \sqrt[3]{\frac{6}{\pi}(t-1) + 125}$

5 a $\frac{dy}{dx} = e^{-xy}$ is not a variable separable equation because it cannot be written in the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

b and c



d At each point (x, y) , $x \neq 0$, $y \neq 0$ the gradients of the tangents to the solutions of each DE are given by $\frac{dy}{dx} = \frac{2y}{x}$ and $\frac{dy}{dx} = -\frac{x}{2y}$.
As $\left(\frac{2y}{x}\right) \times \left(-\frac{x}{2y}\right) = -1$ these tangents are orthogonal at these points.



Short investigation

year	annually	monthly	daily	hourly	continuously
40	4,115.19	4211.61	4220.40	4220.68	4220.70

Exercise 3D

1 The model is $Q(t) = Q_0 e^{-\frac{\ln 2}{5568} t}$
 $Q(t) = 0.55 \cdot Q_0 \Rightarrow Q_0 e^{-\frac{\ln 2}{5568} t} = 0.55 \cdot Q_0$
 $= 0.55 \cdot Q_0 \Rightarrow -\frac{\ln 2}{5568} t = 0.55$
 $\Rightarrow t = 4800$ years

2 The model is $Q(t) = Q_0 e^{-\frac{\ln 2}{5530} t}$
 $Q(t) = 0.55 \cdot Q_0 \Rightarrow Q_0 e^{-\frac{\ln 2}{5530} t} = 0.55 \cdot Q_0$
 $= 0.55 \cdot Q_0 \Rightarrow -\frac{\ln 2}{5530} t = 0.55$
 $\Rightarrow t = 4770$ years

$$3 \text{ a } \frac{dV}{dt} = -0.00015V$$

$$b \quad \frac{dV}{dt} = -0.00015V \Rightarrow \int \frac{dV}{V} = -0.00015 \int dt$$

$$\Rightarrow \ln V = -0.00015t + c$$

$$\therefore V = V_0 e^{-0.00015t}$$

$$V(2) = 0.0008 \Rightarrow V_0 e^{-0.00015 \times 2} = 0.0008$$

$$\Rightarrow V_0 = 0.00080024... < 0.001$$

So, the driver's blood alcohol rate was within the permissible limits.

$$4 \quad \frac{dP}{dt} = 0.0004P(250 - P) \Rightarrow \frac{dP}{P(250 - P)} = \frac{1}{2500} dt$$

$$\int \frac{dP}{P(250 - P)} = \int \frac{1}{2500} dt$$

$$\Rightarrow \frac{1}{250} \int \left(\frac{1}{P} + \frac{1}{250 - P} \right) dP = \frac{1}{2500} \int dt$$

$$\int \left(\frac{1}{P} + \frac{1}{250 - P} \right) dP = \frac{1}{10} \int dt \Rightarrow \ln \left| \frac{P}{250 - P} \right| = \frac{1}{10} t + c$$

$$\left| \frac{P}{250 - P} \right| = A e^{0.1t}$$

$$P(0) = 35 \Rightarrow A = \frac{7}{43}$$

$$\therefore \left| \frac{P}{250 - P} \right| = \frac{7}{43} e^{0.1t} \text{ and } P(t) = 240 \Rightarrow t = 49.93...,$$

i.e., approx. 50 years.

$$5 \text{ a } \frac{dP}{dt} = \frac{0.03}{1+t} P \Rightarrow \frac{dP}{P} = \frac{0.03}{1+t} dt \Rightarrow P = A(1+t)^{0.03}$$

$$P(0) = 100 \Rightarrow A = 100$$

$$\therefore P = 100(1+t)^{0.03} \text{ and } P(10) = 100 \times (11)^{0.03}$$

$$\Rightarrow P(10) = 107.459...$$

$$\therefore 107.46 \text{ euros}$$

$$b \quad P = 100(1+t)^{I_0}$$

$$P(10) = 100 \times 11^{I_0}$$

$$c \quad \frac{dP}{dt} = (0.01 + 0.001t)P \Rightarrow \frac{dP}{P} = (0.01 + 0.001t)dt$$

$$\int \frac{dP}{P} = \int (0.01 + 0.001t)dt$$

$$\Rightarrow \ln |P| = 0.01t + 0.0005t^2 + c$$

$$P(0) = 100 \Rightarrow c = \ln(100)$$

$$P = 100e^{0.01t + 0.0005t^2}$$

$$P(10) = 116.183... = 116.18 \text{ euros}$$

$$6 \text{ a } m \frac{dv}{dt} = mg - kv^2 \Rightarrow \frac{dv}{dt} = g - \frac{k}{m} v^2$$

$$\Rightarrow g \frac{dv}{g^2 - \frac{gk}{m} v^2} = dt$$

$$\alpha = \sqrt{\frac{gk}{m}} \Rightarrow \frac{g}{g^2 - \alpha^2 v^2} dv = dt$$

$$\text{As } \frac{g}{g^2 - \alpha^2 v^2} = \frac{1}{2} \left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v} \right) \text{ the}$$

equation can be written as

$$\left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v} \right) dv = 2dt$$

$$b \quad \int \frac{g}{g^2 - \alpha^2 v^2} dv = \int dt \Rightarrow \int \left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v} \right) dv$$

$$= 2 \int dt - \frac{1}{\alpha} \ln \left| \frac{g + \alpha v}{g - \alpha v} \right| = t^2 + c \Rightarrow \frac{g + \alpha v}{g - \alpha v} = A e^{\frac{1}{\alpha} t^2}$$

$$v = \frac{g}{\alpha} \frac{A e^{\frac{1}{\alpha} t^2} - 1}{A e^{\frac{1}{\alpha} t^2} + 1}$$

$$\lim_{x \rightarrow \infty} v(t) = \frac{g}{\alpha}$$

Exercise 3E

$$1 \text{ a } \frac{dy}{dx} + \underbrace{\frac{3}{x}}_{p(x)} y = \underbrace{e^{2x}}_{q(x)}$$

$$I(x) = e^{\int 3dx} = e^{3x}$$

Multiplying both sides of $\frac{dy}{dx} + 3y = e^x$ by $I(x)$,

$$e^x \cdot \frac{dy}{dx} + 3e^{3x} y = e^{5x} \Rightarrow \frac{d}{dx} (e^{3x} \cdot y) = e^{5x}$$

which is an exact DE

$$\therefore e^{3x} \cdot y = \int e^{5x} dx \Rightarrow e^{3x} \cdot y = \frac{1}{5} e^{5x} + c$$

$$\therefore y = \frac{1}{5} e^{2x} + c \cdot e^{-3x}$$

$$b \quad \frac{dy}{dx} + \underbrace{(2x+1)}_{p(x)} y = \underbrace{e^{-x^2}}_{q(x)}$$

$$I(x) = e^{\int (2x+1)dx} = e^{x^2+x}$$

Multiplying both sides of $\frac{dy}{dx} + (2x+1)y = e^{-x^2}$ by $I(x)$,

$$e^{-x^2} \cdot \frac{dy}{dx} + (2x+1)e^{x^2+x} y = e^x \Rightarrow \frac{d}{dx} (e^{x^2+x} \cdot y) = e^x$$

which is an exact DE

$$\therefore e^{x^2+x} \cdot y = \int e^x dx \Rightarrow e^{x^2+x} \cdot y = e^x + c$$

$$\therefore y = e^{-x^2} + c \cdot e^{-x^2-x}$$

$$\text{c } \frac{dy}{dx} + \underbrace{1}_{p(x)} \cdot y = \underbrace{e^{-2x} \cos(x)}_{q(x)}$$

$$I(x) = e^{\int 1 dx} = e^x$$

Multiplying both sides of $\frac{dy}{dx} + y = e^{-2x} \cos(x)$ by $I(x)$,

$$e^x \frac{dy}{dx} + e^x y = e^{-x} \cos(x) \Rightarrow \frac{d}{dx}(e^x \cdot y) = e^{-x} \cos(x)$$

which is an exact DE

$$\therefore e^x \cdot y = \int e^{-x} \cos(x) dx$$

$$\Rightarrow e^x \cdot y = \frac{1}{2} e^{-x} (\sin(x) - \cos(x)) + c$$

(use integration by parts)

$$\therefore y = \frac{1}{2} e^{-2x} (\sin(x) - \cos(x)) + c \cdot e^{-x}$$

$$y = \frac{1}{2} e^{-x^2} (\sin x - \cos x) + A e^{-x}$$

$$\text{d } y = \left(\frac{x^2}{2} + c \right) \cos x$$

$$\frac{dy}{dx} + \underbrace{\tan(x)}_{p(x)} \cdot y = \underbrace{\cos(x)}_{q(x)}$$

$$I(x) = e^{\int \tan(x) dx} = e^{-\ln(\cos(x))} = \sec(x)$$

Multiplying both sides of $\frac{dy}{dx} + y \tan(x) = \cos(x)$ by $I(x)$,

$$\sec(x) \cdot \frac{dy}{dx} + \tan(x) \sec(x) \cdot y = 1 \Rightarrow \frac{d}{dx}$$

$(\sec(x) \cdot y) = 1$ which is an exact DE

$$\therefore \sec(x) \cdot y = \int 1 dx \Rightarrow \sec(x) \cdot y = x + c \text{ (use}$$

integration by parts)

$$\therefore y = (x + c) \cdot \cos(x)$$

$$\text{e } \frac{dy}{dx} + \underbrace{\cot(x)}_{p(x)} \cdot y = \underbrace{\cos(x)}_{q(x)}$$

$$I(x) = e^{\int \cot(x) dx} = e^{\ln(\sin(x))} = \sin(x)$$

Multiplying both sides of $\frac{dy}{dx} + y \cot(x) = \cos(x)$ by $I(x)$,

$$\sin(x) \frac{dy}{dx} + \cos(x) \cdot y = \sin(x)$$

$$\Rightarrow \frac{d}{dx}(\sin(x) \cdot y) = \sin(x) \text{ which is an exact DE}$$

$$\therefore \sin(x) \cdot y = \int \underbrace{\sin(x) \cos(x)}_{\frac{1}{2} \sin(2x)} dx$$

$$\Rightarrow \sin(x) \cdot y = -\frac{1}{4} \cos(2x) + c$$

$$\therefore y = \left(-\frac{1}{4} \cos(2x) + c \right) \cdot \csc(x)$$

$$\text{2 a } \frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} + y = x$$

$$I(x) = e^{\int 1 dx} = e^x$$

Multiplying both sides of $\frac{dy}{dx} + y = c$ by $I(x)$,

$$e^x \frac{dy}{dx} + e^x y = x \cdot e^x \Rightarrow \frac{d}{dx}(e^x \cdot y) = x \cdot e^x \text{ which is an exact DE}$$

$$\therefore e^x \cdot y = \int x \cdot e^x dx \Rightarrow e^x \cdot y = (x - 1)e^x + c \text{ (use integration by parts)}$$

$$\therefore y = x - 1 + c \cdot e^{-x}$$

$$y(0) = -2 \Rightarrow -1 + c = -2 \Rightarrow c = -1$$

$$\therefore y = x - 1 - e^{-x}$$

$$\text{b } \frac{dy}{dx} - 2y = \sin x$$

$$I(x) = e^{\int -2 dx} = e^{-2x}$$

Multiplying both sides of $\frac{dy}{dx} - 2y = \sin x$ by $I(x)$,

$$e^{-2x} \frac{dy}{dx} - 2e^{-2x} y = e^{-2x} \sin(x)$$

$$\Rightarrow \frac{d}{dx}(e^{-2x} y) = e^{-2x} \sin(x)$$

which is an exact DE

$$\therefore e^{-2x} y = \int e^{-2x} \sin(x) dx$$

$$\Rightarrow e^{-2x} y = \frac{1}{5} (-2 \sin(x) - \cos(x)) e^{-2x} + c$$

$$y(0) = 0 \Rightarrow c = \frac{1}{5}$$

$$y = \frac{1}{5} (-2 \sin(x) - \cos(x) + e^{2x})$$

$$\text{c } \frac{dy}{dx} + (x + 1) y = e^{-\frac{x^2}{2}}$$

$$I(x) = e^{\int (x+1) dx} = e^{\frac{x^2}{2} + x}$$

Multiplying both sides of $\frac{dy}{dx} + (x + 1) y = e^{-\frac{x^2}{2}}$ by $I(x)$,

$$e^{\frac{x^2}{2} + x} \frac{dy}{dx} + e^{\frac{x^2}{2} + x} (x + 1) y = e^x \Rightarrow \frac{d}{dx} \left(e^{\frac{x^2}{2} + x} y \right) = e^x$$

which is an exact DE

$$\therefore e^{\frac{x^2}{2} + x} y = \int e^x dx \Rightarrow e^{\frac{x^2}{2} + x} y = e^x + c$$

$$y(0) = 2 \Rightarrow c = 1$$

$$y = e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2} - x}$$

3 a $x \frac{dy}{dx} + y = x^2 + x \Rightarrow \frac{d}{dx}(xy) = x^2 + x$

$\therefore$ the equation is exact.

$$\frac{d}{dx}(xy) = x^2 + x \Rightarrow xy = \int (x^2 + x) dx$$

$$\Rightarrow xy = \frac{x^3}{3} + \frac{x^2}{2} + c$$

$$\therefore y = \frac{x^2}{3} + \frac{x}{2} + \frac{c}{x}$$

$$x \frac{dy}{dx} + y = x^2 + x \Rightarrow \frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{x+1}{q(x)}$$

$\therefore$ the equation is linear

b $(x+1) \frac{dy}{dx} + y = \sin(x)e^x \Rightarrow \frac{d}{dx}((x+1)y) = \sin(x)e^x$

$\therefore$ the equation is exact.

$$\frac{d}{dx}((x+1)y) = \sin(x)e^x \Rightarrow (x+1)y = \int \sin(x) e^x dx$$

$$\Rightarrow (x+1)y = \frac{1}{2}(\sin x - \cos x) e^x + c$$

$$\therefore y = \frac{\sin x - \cos x}{2(x+1)} e^x + \frac{c}{x+1}$$

$$(x+1) \frac{dy}{dx} + y = \sin(x)e^x \Rightarrow \frac{dy}{dx} + \frac{1}{x+1} \cdot y = \frac{\sin(x)e^x}{x+1}$$

$\therefore$ the equation is linear

c $(\cos x) \frac{dy}{dx} - (\sin x)y = \cos^4 x$

$$\Rightarrow \frac{d}{dx}(\cos(x) y) = \cos^4 x$$

$\therefore$ the equation is exact.

$$\frac{d}{dx}(\cos(x) y) = \cos^4 x \Rightarrow \cos(x) y = \int \cos^4 x dx$$

$$\text{As } \cos^4 x = \left(\frac{1 + \cos(2x)}{2} \right)^2 = \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}$$

$$= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8}(1 + \cos(4x))$$

$$= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

$$\Rightarrow \cos(x)y = \int \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) dx$$

$$\Rightarrow \cos(x)y = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$$

$$\therefore y = \left(\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c \right) \sec(x)$$

$$(\cos x) \frac{dy}{dx} - (\sin x)y = \cos^4 x \Rightarrow \frac{dy}{dx} - \frac{\tan(x)}{p(x)} \cdot y = \frac{\cos^3(x)}{q(x)}$$

$\therefore$ the equation is linear

d $(\sin(x)) \frac{dy}{dx} + (\cos(x))y = \tan(x)$

$$\Rightarrow \frac{d}{dx}(\sin(x)y) = \tan(x)$$

$\therefore$ the equation is exact.

$$\frac{d}{dx}(\sin(x)y) = \tan(x) \Rightarrow \sin(x)y = \int \tan x dx$$

$$\Rightarrow \sin(x)y = -\ln|\cos(x)| + c$$

$$\Rightarrow \sin(x)y = -\ln|\cos x| + c$$

$$\therefore y = \frac{c - \ln|\cos x|}{\sin x} \text{ or } y = (\ln|\sec(x)| + c) \csc(x)$$

$$(\sin(x)) \frac{dy}{dx} + (\cos(x))y = \tan(x)$$

$$\Rightarrow \frac{dy}{dx} + \frac{\cot(x)}{p(x)} \cdot y = \frac{\sec(x)}{q(x)}$$

$\therefore$ the equation is linear

4 a $x^2 \frac{dy}{dx} - x^2 y = y \Rightarrow \frac{dy}{dx} - (1 + x^{-2})y = 0$

$\therefore$ the equation is linear

$$I(x) = e^{\int -1 - x^{-2} dx} = e^{-x + \frac{1}{x}}$$

Multiplying both sides of $\frac{dy}{dx} - (1 + x^{-2})y = 0$ by $I(x)$,

$$e^{-x + \frac{1}{x}} \frac{dy}{dx} - (1 + x^{-2}) e^{-x + \frac{1}{x}} y = 0 \Rightarrow \frac{d}{dx} \left(e^{-x + \frac{1}{x}} y \right) = 0$$

which is an exact DE

$$e^{-x + \frac{1}{x}} y = A \Rightarrow y = A e^{\frac{1}{x} - x}$$

$$\frac{dy}{dx} - y = x^{-2} y \Rightarrow \frac{dy}{dx} = (x^{-2} + 1)y$$

$$\Rightarrow \frac{dy}{y} = (x^{-2} + 1)dx$$

$\therefore$ the equation is variable separable

$$\int \frac{dy}{y} = \int (x^{-2} + 1) dx \Rightarrow \ln|y| = -\frac{1}{x} + x + c$$

$$\therefore y = A e^{\frac{1}{x} - x}$$

b $(x^2 + 1) \frac{dy}{dx} - xy = 0 \Rightarrow \frac{dy}{dx} - \frac{x}{x^2 + 1} y = 0$

$\therefore$ the equation is linear

$$I(x) = e^{\int -\frac{x}{x^2 + 1} dx} = e^{-\frac{1}{2} \ln(x^2 + 1)} = \frac{1}{\sqrt{x^2 + 1}}$$

Multiplying both sides of $\frac{dy}{dx} - \frac{x}{x^2 + 1} y = 0$ by $I(x)$,

$$\frac{1}{\sqrt{x^2 + 1}} \frac{dy}{dx} - \frac{x}{x^2 + 1} \frac{1}{\sqrt{x^2 + 1}} y = 0$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{\sqrt{x^2 + 1}} y \right) = 0$$

which is an exact DE

$$\frac{1}{\sqrt{x^2+1}} y = 0 = A \Rightarrow y = A\sqrt{x^2+1}$$

$$\frac{dy}{dx} - \frac{x}{x^2+1} y = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1} y$$

$$\Rightarrow \frac{dy}{y} = \frac{x}{x^2+1} dx$$

$\therefore$ the equation is variable separable

$$\int \frac{dy}{y} = \int \frac{x}{x^2+1} dx \Rightarrow \ln|y| = \frac{1}{2} \ln(x^2+1) + c$$

$$\therefore y = A\sqrt{x^2+1}$$

c $\frac{1}{x} \frac{dy}{dx} + 2y = 3 \Rightarrow \frac{dy}{dx} + 2xy = 3x$

$\therefore$ the equation is linear

$$I(x) = e^{\int 2x dx} = e^{x^2}$$

Multiplying both sides of $\frac{dy}{dx} + 2xy = 3x$ by $I(x)$,

$$e^{x^2} \frac{dy}{dx} + 2xe^{x^2} y = 3xe^{x^2} \Rightarrow \frac{d}{dx}(e^{x^2} y) = 3xe^{x^2}$$

which is an exact DE

$$e^{x^2} y = \frac{3}{2} e^{x^2} + c \Rightarrow y = \frac{3}{2} + Ae^{-x^2}$$

$$\frac{dy}{dx} + 2xy = 3x \Rightarrow \frac{dy}{dx} = x(3-2y)$$

$$\Rightarrow \frac{dy}{3-2y} = x dx$$

$\therefore$ the equation is variable separable

$$\int \frac{dy}{3-2y} = \int x dx \Rightarrow -\frac{1}{2} \ln|3-2y| = \frac{1}{2} x^2 + c$$

$$\int \frac{dy}{3-2y} = \int x dx \Rightarrow -\frac{1}{2} \ln|3-2y| = \frac{1}{2} x^2 + c$$

$$\Rightarrow \ln|3-2y| = -x^2 + c$$

$$\Rightarrow 3-2y = e^{-x^2+c}$$

$$\Rightarrow 2y = 3 - Ae^{-x^2}$$

$$\therefore y = \frac{3}{2} + Be^{-x^2}$$

5 $u = \frac{1}{y} \Rightarrow \frac{dy}{dx} = -\frac{1}{u^2} \frac{du}{dx}$

$$\frac{dy}{dx} + xy(1-y) = 0 \Rightarrow -\frac{1}{u^2} \frac{du}{dx} + x \frac{1}{u} \left(1 - \frac{1}{u}\right) = 0$$

$$\Rightarrow \frac{du}{dx} - xu = -x \text{ which is a linear equation of the}$$

$$\text{form } \frac{du}{dx} + p(x)u = q(x).$$

$$I(x) = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

Multiplying both sides of $\frac{du}{dx} - xu = -x$ by $I(x)$,

$$e^{-\frac{x^2}{2}} \frac{du}{dx} - xe^{-\frac{x^2}{2}} u = -xe^{-\frac{x^2}{2}} \Rightarrow \frac{d}{dx} \left(e^{-\frac{x^2}{2}} u \right) = -xe^{-\frac{x^2}{2}}$$

which is an exact DE

$$e^{-\frac{x^2}{2}} u = e^{-\frac{x^2}{2}} + A \Rightarrow u = 1 + Ae^{\frac{x^2}{2}}$$

$$\frac{1}{y} = 1 + Ae^{\frac{x^2}{2}} \Rightarrow y = \frac{1}{1 + Ae^{\frac{x^2}{2}}}$$

$$y(1) = 2 \Rightarrow 2 = \frac{1}{1 + Ae^{\frac{1}{2}}} \Rightarrow A = -\frac{1}{2} e^{-\frac{1}{2}}$$

$$y = \frac{1}{1 - \frac{1}{2} e^{\frac{x^2}{2} - \frac{1}{2}}}$$

Exercise 3F

- 1 Use cooling model equation $T(t) = ke^{\alpha t} + T_e$ and solve simultaneously the

$$\text{equations: } \begin{cases} T(0) = 24 \\ T(5) = 22 \\ T(10) = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} k + T_e = 24 \\ ke^{5\alpha} + T_e = 22 \\ ke^{10\alpha} + T_e = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k(e^{5\alpha} - e^{10\alpha}) = 1.6 \\ ke^{10\alpha} + T_e = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k = \frac{1.6}{e^{5\alpha} - e^{10\alpha}} \\ ke^{10\alpha} + 24 - k = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k = \frac{1.6}{e^{5\alpha} - e^{10\alpha}} \\ \frac{1.6}{e^{5\alpha} - e^{10\alpha}} (e^{10\alpha} - 1) + 24 = 20.4 \end{cases}$$

to obtain $\alpha = -0.044628\dots$, $k = 10$ and $T_e = 14$.

Exercise 3F	
$\text{nSolve}\left(\frac{1.6}{e^{5x}-e^{10x}}(e^{10x}-1)+24=20.4, x\right)$	
-0.04462871	
$d:=-0.044628710262786$	-0.04462871
$k:=\frac{1.6}{e^{5a}-e^{10a}}$	$10.$
$t:=24-k$	$14.$
	$8/99$

So the temperature outside is approximately
14 degrees

2 a $a = 5000, b = 0.04$

b $x(0) = 1000$

$$\frac{dx}{dt} = 5000 + 0.04x \Rightarrow \int \frac{dx}{5000 + 0.04x} = \int dt$$

$$\therefore \frac{1}{0.04} \ln |5000 + 0.04x| = t + c$$

$$\Rightarrow 5000 + 0.04x = Ae^{0.04t}$$

$$x(0) = 1000 \Rightarrow A = 5040$$

$$x = 126000e^{0.04t} - 125000$$

$$\text{or } x = 1000e^{0.04t} + 125000(e^{0.04t} - 1)$$

c $x(t) = 100000 \Rightarrow t = 14.5$ years.

$x(t) = 1000000 \Rightarrow t = 54.7$ year. So not in the first 50 years.

Exercise 3F	
$f1(x) := 126000 \cdot e^{0.04x} - 125000$	Done
$nSolve(f1(x)=100000, x)$	14.495462
$nSolve(f1(x)=1000000, x)$	54.73141
	3/99

3 a Use $v = \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t}\right)$ as deduced in example 16

with $k = 0.5$ and $m = 90$

For $g = 10, v = 97 \text{ ms}^{-1}$

Note: For $g = 9.8, v = 95.3 \text{ ms}^{-1}$

Exercise 3F	
$v(t) := \frac{90 \cdot 9.8}{0.5} \cdot \left(1 - e^{-\frac{0.5}{90}t}\right)$	Done
$v(10)$	95.327497
$v(t) := \frac{90 \cdot 10}{0.5} \cdot \left(1 - e^{-\frac{0.5}{90}t}\right)$	Done
$v(10)$	97.272956
	4/99

b $\int_0^1 \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t}\right) dt = 491 \text{ metres}$

Note: when $g = 9.8, s = 481 \text{ metres}$

Exercise 3F	
$v(t) := \frac{90 \cdot 10}{0.5} \cdot \left(1 - e^{-\frac{0.5}{90}t}\right)$	Done
$v(10)$	97.272956
$\int_0^{10} v(t) dt$	490.86793
	5/99

c $\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t}\right) = \frac{mg}{k},$

i.e., 90 ms^{-1} as $k = 10$

Note: when $g = 9.8, v = 88.2$

4 a Since solution is flowing out at 10 l/m and the flow rate in is 5 l/m the quantity of water in the tank at any given instant will be $100 - 5t$

$$\frac{dQ}{dt} = 10 \times 5 - \frac{Q}{100 - 5t} \times 10 \Rightarrow \frac{dQ}{dt} + \frac{2Q}{20 - t} = 50$$

$$I(t) = e^{\int \frac{2}{20-t} dt} = e^{-2 \ln |20-t|} = \frac{1}{(20-t)^2}$$

$$\frac{1}{(20-t)^2} \frac{dQ}{dt} - \frac{2}{(20-t)^3} Q = \frac{50}{(20-t)^2}$$

$$\frac{d}{dt} \left(\frac{1}{(20-t)^2} Q \right) = \frac{50}{(20-t)^2}$$

$$\frac{1}{(20-t)^2} Q = \int \frac{50}{(20-t)^2} dt$$

$$\Rightarrow \frac{1}{(20-t)^2} Q = \frac{50}{20-t} + c$$

$$\therefore Q(t) = 50(20-t) + c(20-t)^2$$

$$Q(0) = 0 \Rightarrow c = -2.5$$

$$\therefore Q(t) = 50(20-t) - 2.5(20-t)^2$$

b $Q(10) = 250 \text{ grams (3s.f.)}$

Exercise 3G

1 a $\frac{dy}{dx} = \frac{3x^2 - xy}{x^2} \Rightarrow \frac{dy}{dx} = 3 - \underbrace{\frac{y}{x}}_{f\left(\frac{y}{x}\right)}$ $\therefore$ The DE is

homogeneous.

b $2x \frac{dy}{dx} = x - y + 3 \Rightarrow \frac{dy}{dx} = \frac{1}{2} - \frac{y}{2x} + \frac{3}{2x}$
 $\neq f\left(\frac{y}{x}\right)$

$\therefore$ The DE is non-homogeneous

c $(x^2 + y) \frac{dy}{dx} = x \Rightarrow \frac{dy}{dx} = \frac{x}{x^2 + y} \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{x}}{1 + \frac{y}{x}}$
 $\neq f\left(\frac{y}{x}\right)$

$\therefore$ The DE is non-homogeneous

2 a $\frac{dy}{dx} = \frac{3x^2 - xy}{x^2} \Rightarrow \frac{dy}{dx} = 3 - \underbrace{\frac{y}{x}}_{f\left(\frac{y}{x}\right)}$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = 3 - v \Rightarrow x \frac{dv}{dx} = 3 - 2v \text{ which is}$$

variables separable DE

$$\int \frac{dv}{3-2v} = \int \frac{dx}{x} \Rightarrow -\frac{1}{2} \ln|3-2v| = \ln|x| + c$$

$$\therefore -\frac{1}{2} \ln\left|3 - \frac{2y}{x}\right| = \ln|x| + c \text{ or } y = \frac{x}{2} \left(3 + \frac{A}{x^2}\right)$$

3 a $2x \frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} = \frac{x-y}{2x} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - \frac{1}{2} \cdot \frac{y}{x}$

(homogeneous)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

Method 1:

$$v + x \frac{dv}{dx} = \frac{1}{2} - \frac{1}{2} v \Rightarrow x \frac{dv}{dx} = \frac{1}{2} - \frac{3}{2} v$$

(variable separable)

$$\int \frac{2}{1-3v} dv = \int \frac{dx}{x} \Rightarrow -\frac{2}{3} \ln|1-3v| = \ln|x| + c$$

$$\therefore (1-3v)^{-\frac{2}{3}} = Ax \Rightarrow \left(1 - \frac{3y}{x}\right)^{-\frac{2}{3}} = Ax$$

Method 2:

$$v + x \frac{dv}{dx} = \frac{1}{2} - \frac{1}{2} v \Rightarrow \frac{dv}{dx} + \frac{3}{2x} v = \frac{1}{2x}$$

(linear DE)

$$I(x) = e^{\int \frac{3}{2x} dx} = e^{\frac{3}{2} \ln(x)} = x^{\frac{3}{2}} \text{ or } \sqrt{x^3}$$

Multiplying both sides of $\frac{dv}{dx} + \frac{3}{2x} v = \frac{1}{2x}$ by $I(x)$,

$$x^{\frac{3}{2}} \frac{dv}{dx} + \frac{3}{2} x^{\frac{1}{2}} v = \frac{1}{2} x^{\frac{1}{2}} \Rightarrow \frac{d}{dx} \left(x^{\frac{3}{2}} v \right) = \frac{1}{2} x^{\frac{1}{2}}$$

$$\therefore x^{\frac{3}{2}} v = \frac{1}{3} x^{\frac{3}{2}} + c \Rightarrow v = \frac{1}{3} + cx^{-\frac{3}{2}} \Rightarrow y = \frac{1}{3} x + cx^{-\frac{1}{2}}$$

Use two different methods to solve the following differential equations

b $x \frac{dy}{dx} = 2x + 3y$

$$x \frac{dy}{dx} = 2x + 3y \Rightarrow \frac{dy}{dx} = \frac{2x+3y}{x} \Rightarrow \frac{dy}{dx} = 2 + 3 \cdot \frac{y}{x}$$

with final solution $y = Ax^3 - x$

Method 1:

$$v + x \frac{dv}{dx} = 2 + 3v$$

$$\Rightarrow x \frac{dv}{dx} = 2 + 2v$$

$$\Rightarrow \frac{1}{2} \int \frac{dv}{1+v} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \ln|1+v| = \ln|x| + c$$

$$\Rightarrow \ln|1+v| = \ln Ax^2$$

$$\Rightarrow 1 + \frac{y}{x} = Ax^2$$

$$\Rightarrow y = Ax^3 - x$$

Method 2:

$$v + x \frac{dv}{dx} = 2 + 3v$$

$$\Rightarrow x \frac{dv}{dx} = 2 + 2v$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x} \cdot v = \frac{2}{x}$$

$$I = e^{\int -\frac{2}{x} dx} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$\frac{1}{x^2} \frac{dv}{dx} - \frac{2}{x^3} \cdot v = \frac{2}{x^3}$$

$$\Rightarrow \frac{d}{dx} \left(\frac{v}{x^2} \right) = \frac{2}{x^3}$$

$$\Rightarrow \frac{v}{x^2} = \int \frac{2}{x^3} dx$$

$$\Rightarrow \frac{v}{x^2} = -\frac{1}{x^2} + c$$

$$\Rightarrow y = -x + cx^3$$

4 a $x^2 \frac{dy}{dx} = 4x^2 + y^2 \Rightarrow \frac{dy}{dx} = \frac{4x^2 + y^2}{x^2}$

$$\Rightarrow \frac{dy}{dx} = 4 + \underbrace{\left(\frac{y}{x}\right)^2}_{f\left(\frac{y}{x}\right)} \text{ (homogeneous)}$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = 4 + v^2 \Rightarrow x \frac{dv}{dx} = 4 - v + v^2$$

(variable separable)

$$4 - v + v^2 = \left(v - \frac{1}{2}\right)^2 + \frac{15}{4}$$

$$\int \frac{1}{4 - v - v^2} dv = \int \frac{dx}{x} \Rightarrow \int \frac{1}{\left(v - \frac{1}{2}\right)^2 + \frac{15}{4}} dv = \int \frac{dx}{x}$$

$$\frac{2}{\sqrt{15}} \arctan\left(\frac{2}{\sqrt{15}}\left(\frac{y}{x} - \frac{1}{2}\right)\right) = \ln|x| + c$$

$$\text{b } (x^2 + y^2) \frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 + y^2} \Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2}$$

(homogeneous)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = \frac{v}{1 + v^2} \Rightarrow x \frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$\Rightarrow \int \frac{1 + v^2}{v^3} dv = - \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2v^2} + \ln|v| = -\ln|x| + c$$

$$\Rightarrow \frac{1}{2v^2} = \ln|v| + \ln|x| + c$$

$$\Rightarrow \frac{1}{2v^2} = \ln A|vx|$$

$$\Rightarrow Avx = e^{\frac{1}{2v^2}}$$

$$\Rightarrow Ay = e^{\frac{x^2}{2y^2}}$$

$$\text{c } x(x^2 + y^2) \frac{dy}{dx} = x^3 + 2x^2y + y^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 + 2x^2y + y^3}{x^3 + xy^2}$$

$$\frac{dy}{dx} = \frac{1 + 2 \cdot \frac{y}{x} + \left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)^2} \quad \text{(homogeneous)}$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = \frac{1 + 2v + v^3}{1 + v^2} \Rightarrow x \frac{dv}{dx} = \frac{1 + v}{1 + v^2}$$

(variable separable)

$$\int \frac{1 + v^2}{1 + v} dv = \int \frac{dx}{x} \Rightarrow \left(\int v - 1 + \frac{2}{1 + v} \right) dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}v^2 - v + 2\ln|v + 1| = \ln|x| + c$$

$$\therefore (1 + v)^2 e^{\frac{1}{2}v^2 - v} = Ax \Rightarrow \left(1 + \frac{y}{x} \right)^2 e^{\frac{y^2 - 2xy}{2x^2}} = Ax$$

$$\text{5 a } (x^2 - y^2) \frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 - y^2}$$

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{1 - \left(\frac{y}{x}\right)^2}$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = \frac{v}{1 - v^2} \Rightarrow x \frac{dv}{dx} = \frac{v^3}{1 - v^2}$$

(variable separable)

$$\int \frac{1 - v^2}{v^3} dv = \int \frac{dx}{x} \Rightarrow \left(\int \frac{1}{v^3} - \frac{1}{v} \right) dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2v^2} - \ln|v| = \ln|x| + c$$

$$v(4) = \frac{1}{2} \Rightarrow -2 - \ln \frac{1}{2} = \ln 4 + c \Rightarrow c = -2 - \ln 2$$

$$\therefore -\frac{1}{2v^2} - \ln|v| = \ln|x| - \frac{1}{8} - \ln 2$$

$$\therefore -\frac{1}{2} \left(\frac{x}{y} \right)^2 - \ln \left| \frac{y}{x} \right| - \ln|x| + 2 + \ln 2 = 0 \text{ or}$$

$$-\frac{1}{2} \left(\frac{x}{y} \right)^2 - \ln|y| + 2 + \ln 2 = 0$$

$$\text{b } x^2 \frac{dy}{dx} = x^2 - xy + y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = 1 - v + v^2 \Rightarrow x \frac{dv}{dx} = 1 - 2v + v^2$$

(variable separable)

$$\int \frac{1}{(1 - v)^2} dv = \int \frac{dx}{x} \Rightarrow -\frac{1}{1 - v} = \ln|x| + c$$

$$v(1) = 0 \Rightarrow c = -1$$

$$\frac{1}{v - 1} = \ln|x| - 1 \Rightarrow v = \frac{\ln|x|}{\ln|x| - 1}$$

$$\therefore y = \frac{x \ln|x|}{\ln|x| - 1}$$

$$\text{c } x^2 \frac{dy}{dx} = x^2 + xy + y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = 1 + v + v^2 \Rightarrow x \frac{dv}{dx} = 1 + v^2$$

(variable separable)

$$\int \frac{1}{1 + v^2} dv = \int \frac{dx}{x} \Rightarrow \arctan(v) = \ln|x| + c$$

$$v(1) = 0 \Rightarrow c = 0$$

$$\therefore \arctan(v) = \ln|x|$$

$$\therefore \arctan\left(\frac{y}{x}\right) = \ln|x|$$



b $\frac{dy}{dx} = \frac{2y^2 - x^2}{3xy}$

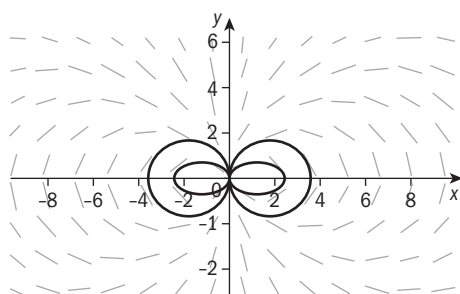
As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = \frac{2v^2 - 1}{3v} \Rightarrow x \frac{dv}{dx} = \frac{-v^2 - 1}{3v}$$

(variable separable)

$$\int \frac{3v}{v^2 + 1} dv = -\int \frac{dx}{x} \Rightarrow -\frac{3}{2} \ln(v^2 + 1) = -\ln|x| + c$$

$$\therefore \left(\frac{y}{x}\right)^2 + 1 = \frac{A}{\sqrt[3]{x^2}}$$



7 (extension)

- a
- terms have different degrees $\therefore$ non-homogeneous DE
 - the equation contains term in y^2 $\therefore$ non-linear DE
 - it is not possible to re-arrange the equation to the form $\frac{dy}{dx} = \frac{f(y)}{g(x)}$ $\therefore$ non-separable DE

b $v = y^2 \Rightarrow y = v^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{dv}{dv} \cdot \frac{dv}{dx} = \frac{1}{2} v^{-\frac{1}{2}} \frac{dv}{dx}$

$$\therefore 2xy \frac{dy}{dx} = 1 + y^2 - x^2 \Rightarrow x \underbrace{v^{\frac{1}{2}} v^{-\frac{1}{2}}}_1 \frac{dv}{dx} = 1 + v - x^2$$

$$\therefore x \frac{dv}{dx} = 1 + v - x^2 \text{ QED}$$

c $u = x^2 \Rightarrow x = u^{\frac{1}{2}}$ and $\frac{dv}{dx} = \frac{dv}{du} \frac{du}{dx} \Rightarrow \frac{dv}{dx} = 2x \frac{dv}{du}$

$$\therefore x \frac{dv}{dx} = 1 + v - x^2 \Rightarrow 2u \frac{dv}{du} = 1 + v - u$$

$$\therefore 2u \frac{dv}{du} = 1 + v - u \text{ QED}$$

d $w = 1 + v \Rightarrow v = 1 - w$ and

$$\frac{dw}{du} = \frac{dw}{dv} \frac{dv}{du} \Rightarrow \frac{dw}{du} = \frac{dv}{du}$$

$$\therefore 2u \frac{dv}{du} = 1 + v - u \Rightarrow 2u \frac{dw}{du} = w - u \text{ QED}$$

e $2u \frac{dw}{du} = w - u \Rightarrow \frac{dw}{du} = \frac{1}{2} \cdot \frac{w}{u} - 1$ (homogeneous)

$$2u \frac{dw}{du} = w - u \Rightarrow \frac{dw}{du} - \frac{1}{2u} w = -\frac{1}{2} \text{ (linear)}$$

f $2u \frac{dw}{du} = w - u \Rightarrow \frac{dw}{du} = \frac{w - u}{2u} \Rightarrow \frac{dw}{du} = \frac{1}{2} \left(\frac{w}{u} - 1 \right)$

As $w = ut \Rightarrow \frac{dw}{du} = t + u \frac{dt}{du}$, substituting in the equation above,

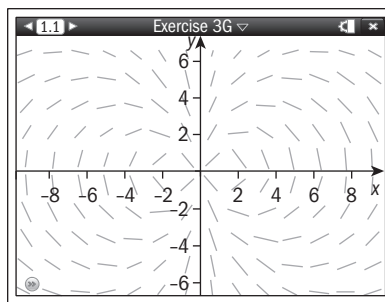
$$t + u \frac{dt}{du} = \frac{t - 1}{2} \Rightarrow u \frac{dt}{du} = -\frac{t + 1}{2}$$

(variable separable)

$$\int \frac{dt}{t + 1} = -\frac{1}{2} \int \frac{1}{u} du \Rightarrow \ln|t + 1| = -\frac{1}{2} \ln|x| + c$$

$$t + 1 = \frac{A}{\sqrt{u}} \Rightarrow \frac{w}{u} + 1 = \frac{A}{\sqrt{u}} \Rightarrow \frac{v + 1}{u} + 1 = \frac{A}{\sqrt{u}}$$

$$\therefore \frac{y^2 + 1}{x^2} + 1 = \frac{A}{x} \Rightarrow x^2 + y^2 - Ax + 1 = 0 \text{ circles}$$



8 (extension)

a $x = u - 1; y = v + 1 \Rightarrow \frac{dv}{du} = \frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{x - y + 2}{x + y} \Rightarrow \frac{dv}{du} = \frac{u - v}{u + v} \Rightarrow \frac{dv}{du} = \frac{1 - \frac{v}{u}}{1 + \frac{v}{u}}$$

(homogeneous)

b As $v = ut \Rightarrow \frac{dv}{du} = t + u \frac{dt}{du}$, substituting in the equation above,

$$t + u \frac{dt}{du} = \frac{1 - t}{1 + t} \Rightarrow u \frac{dt}{du} = \frac{1 - 2t - t^2}{1 + t}$$

(variable separable)

$$\int \frac{2t+2}{t^2+2t-1} dt = -2 \int \frac{1}{u} du$$

$$\Rightarrow \ln|t^2+2t-1| = -2 \ln|u| + c$$

$$\therefore t^2+2t-1 = \frac{A}{u^2} \Rightarrow \left(\frac{y-1}{x+1}\right)^2 + 2\left(\frac{y-1}{x+1}\right) - 1 = \frac{A}{(x+1)^2}$$

$$\therefore (y-1)^2 + 2(y-1)(x+1) - (x+1)^2 = A$$

9 (extension)

a $v = \frac{1}{y}; \frac{dy}{dx} + 2y = y^2 \Rightarrow -\frac{1}{v^2} \frac{dv}{dx} + \frac{2}{v} = \frac{1}{v^2}$

$$\therefore \frac{dv}{dx} - 2v = -1 \text{ (linear DE)}$$

$$I(x) = e^{\int -2dx} = e^{-2x}$$

Multiplying both sides of $\frac{dv}{dx} - 2v = -1$ by $I(x)$

$$e^{-2x} \frac{dv}{dx} - 2e^{-2x}v = -e^{-2x} \Rightarrow \frac{d}{dx}(e^{-2x}v) = -e^{-2x}$$

$$e^{-2x}v = -\frac{1}{2}e^{-2x} + c \Rightarrow v = -\frac{1}{2} + c \cdot e^{2x}$$

$$\therefore y = \frac{2}{1 + Ae^{2x}}$$

b $v = \frac{1}{y^2}; \frac{dy}{dx} - y = y^3 \Rightarrow -v^{-\frac{3}{2}} \frac{dv}{dx} - v^{-\frac{1}{2}} = v^{-\frac{3}{2}}$

$$\therefore \frac{dv}{dx} + 2v = -2 \text{ (linear DE)}$$

$$I(x) = e^{\int 2dx} = e^{2x}$$

Multiplying both sides of $\frac{dv}{dx} + 2v = -2$ by $I(x)$

$$e^{2x} \frac{dv}{dx} + 2e^{2x}v = -2e^{2x} \Rightarrow \frac{d}{dx}(e^{2x}v) = -2e^{2x}$$

$$e^{2x}v = -e^{2x} + A \Rightarrow v = -1 + Ae^{-2x}$$

$$\therefore y^2 = \frac{1}{Ae^{-2x} - 1}$$

Investigation - Euler's Number

e.g. approximations for $h = 0.1, 0.01$ and 0.001 ; the approximation of e gets better

Exercise 3G ▾					
1.1	1.2	A	B	C	D
		n	xn	yn	f
◆			=0+'n*0.1		
7		6	0.6	1.77156	1.77156
8		7	0.7	1.94872	1.94872
9		8	0.8	2.14359	2.14359
10		9	0.9	2.35795	2.35795
11		10	1.	2.59374	2.59374
C11		=c10+0.1*d10			

1.1		1.2		1.3		Exercise 3G									
A		n		B		x _n		C		y _n		D		f	
◆										=0+ 'n*0.01					
97		96				0.96		2.59927		2.59927					
98		97				0.97		2.62527		2.62527					
99		98				0.98		2.65152		2.65152					
100		99				0.99		2.67803		2.67803					
101		100				1.		2.70481		2.70481					
C101								=c100+0.01*d100							

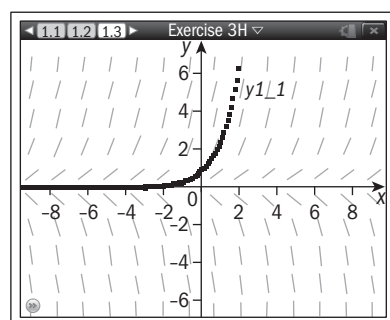
1.1		1.2		1.3		Exercise 3G			
		B		C		3		T	
◆				=0+'n*0.001					
997	996	0.996		2.70608		2.70608			
998	997	0.997		2.70879		2.70879			
999	998	0.998		2.7115		2.7115			
100	999	0.999		2.71421		2.71421			
1001	1000	1.		2.71692		2.71692			
C1001		=c1000+0.001*d1000							

and for $h = 0.0001$, $y(1) = 2.7105461...$ and this is a worse approx. of e

Exercise 3H

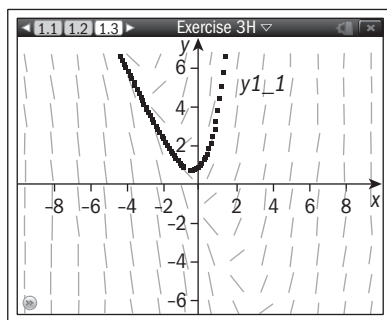
1

1.1		1.2		Exercise 3H			
A	n	B	x _n	C	y _n	D	f
◆				=0+'n*0.1			
7	6		0.6	1.77156	1.77156		
8	7		0.7	1.94872	1.94872		
9	8		0.8	2.14359	2.14359		
10	9		0.9	2.35795	2.35795		
11	10		1.	2.59374	2.59374		
C11 =c10+0.1*d10							



$$y(1) = 2.59$$

2 a, b and d



	A	B	C	D
	xn	yn	f	
997	996	0.996	4.12625	6.11825
998	997	0.997	4.13237	6.12637
999	998	0.998	4.13849	6.13449
100	999	0.999	4.14463	6.14263
1001	1000	1.	4.15077	6.15077

$y(1) = 4.1507...$ when $h = 0.001$

e $\frac{dy}{dx} = 2x + y \Rightarrow \frac{dy}{dx} - y = 2x$ (linear DE)

$I(x) = e^{\int -1 dx} = e^{-x}$

Multiplying both sides of $\frac{dy}{dx} - y = 2x$ by $I(x)$,

$e^{-x} \frac{dy}{dx} - e^{-x} y = 2xe^{-x} \Rightarrow \frac{d}{dx}(e^{-x} y) = 2xe^{-x}$

which is an exact DE

$\therefore e^{-x} y = 2 \int xe^{-x} dx \Rightarrow e^{-x} y = -2(x+1)e^{-x} + c$

$\therefore y = -2(x+1) + c \cdot e^x$

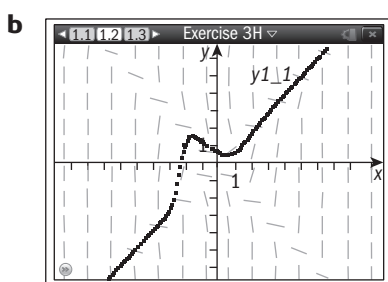
$y(0) = 1 \Rightarrow c = 2$

$\therefore y = -2(x+1) + 2e^x$ and $y(1) = -4 + 2e$

f The approximation gives the correct answer to 3 s.f.

	A	B	C	D
	x	y	f	
2	-1.8	1.6	0.68	
3	-1.6	1.736	-0.453696	
4	-1.4	1.64526...	-0.74688...	
5	-1.2	1.49588...	-0.79766...	
6	-1.	1.33635...	-0.78583...	

$y(-1) = 1.34$



4 a

	A	B	C	D
	xn	yn	f	
7	2.6	1.16285...	0.21037...	
8	2.7	1.18389...	0.19593...	
9	2.8	1.20349...	0.18243...	
10	2.9	1.22173...	0.16980...	
11	3.	1.23871...	0.15799...	

$y = 1.24$

b $\frac{dy}{dx} = \frac{x}{\sin y \times e^x}$

$\Rightarrow \int \sin y dy = \int xe^{-x} dx$

$\Rightarrow \cos y = e^{-x}(x+1) + c$ (integrating RHS by parts)

$y(2) = 1 \Rightarrow c = \cos 1 - 3e^{-2}$

$\therefore y = \arccos(e^{-x}(x+1) + \cos 1 - 3e^{-2})$

$\Rightarrow y(3) = 1.2308...$

$\therefore$ exact value is $y = 1.23$

c e.g. Graphical method not reliable to predict values which are not close to initial value.

5 a

	A	B	C	D
	xn	yn	f	
7	2.6	0.62902...	-0.42151...	
8	2.7	0.58687...	-0.38228...	
9	2.8	0.54864...	-0.34755...	
10	2.9	0.51389...	-0.31674...	
11	3.	0.48221...	-0.28932...	

$y = 0.48$

b $\frac{dy}{dx} = \frac{-2xy}{1+x^2}$ (separable)

$\Rightarrow \int \frac{dy}{y} = - \int \frac{2x}{1+x^2} dx$

$\therefore \ln|y| = -\ln(1+x^2) + c \Rightarrow y = \frac{A}{1+x^2}$

$y(2) = 1 \Rightarrow A = 5$

$\therefore f(x) = \frac{5}{1+x^2}$

c $f(3) = \frac{5}{1+3^2} = 0.5 > 0.48$.

So the approximation gives the correct value to 1 d.p.

6 a

	A	B	C	D
	xn	yn	f	
1	0.	1.	1.	
2	0.25	1.25	1.5625	
3	0.5	1.640625	2.69165...	
4	0.75	2.31353...	5.35245...	

b $\frac{dy}{dx} = y^2 \Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + c$

$\therefore y = -\frac{1}{x+c}$

$y(0) = 1 \Rightarrow c = -1$

$\therefore y = \frac{1}{1-x}$

$y(0.25) = 1.3, y(0.5) = 2$ and $y(0.75) = 4$

c The error increased as the value of x increased.

7 a The rate of decrease of temperature of a body is proportional to the fourth power of the temperature of the body.

b

	xn	yn	f
1	0.	35.	-6.703125
2	0.25	33.3242...	-5.36609...
3	0.5	31.9826...	-4.43154...
4	0.75	30.8748...	-3.74346...
5	1.	29.9389...	-3.21712...

B5 = b4 + 0.25 * c4

$T(1) = 29.9$

c $\ln \frac{T+M}{T-M} + 2 \arctan \frac{T}{M} - 4M^3 kt = c$

$$\left(\frac{1}{T+M} - \frac{1}{T-M} + \frac{\frac{2}{M}}{1 + \left(\frac{T}{M}\right)^2} \right) \frac{dT}{dt} - 4M^3 k = 0$$

$$\Rightarrow \left(\frac{-2M}{T^2 - M^2} + \frac{2M}{M^2 + T^2} \right) \frac{dT}{dt} - 4M^3 k = 0$$

$$\Rightarrow \left(\frac{-1}{T^4 - M^4} \right) \frac{dT}{dt} - k = 0$$

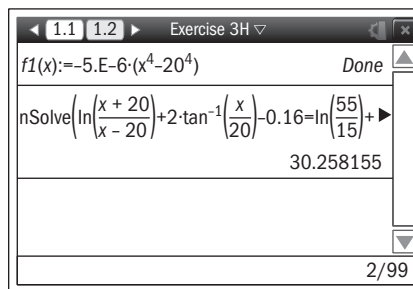
$$\therefore \frac{dT}{dt} = -4k(T^4 - M^4) \text{ QED}$$

$$\ln \left(\frac{T+20}{T-20} \right) + 2 \arctan \left(\frac{T}{20} \right) - \frac{4}{25} t = c$$

$$T(0) = 35 \Rightarrow c = \ln \left(\frac{11}{3} \right) + 2 \arctan \left(\frac{7}{4} \right)$$

$$\begin{aligned} \ln \left(\frac{T+20}{T-20} \right) + 2 \arctan \left(\frac{T}{20} \right) - \frac{4}{25} t \\ = \ln \left(\frac{11}{3} \right) + 2 \arctan \left(\frac{7}{4} \right) \end{aligned}$$

$$\begin{aligned} t = 1 \Rightarrow \ln \left(\frac{T+20}{T-20} \right) + 2 \arctan \left(\frac{T}{20} \right) - \frac{4}{25} \\ = \ln \left(\frac{11}{3} \right) + 2 \arctan \left(\frac{7}{4} \right) \end{aligned}$$



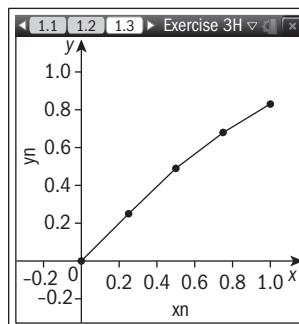
$T(1) = 30.3$

d decrease the step (e.g. for $h = 0.1, T(1) = 30.1$)

	xn	yn	f
8	0.7	31.2614...	-3.975387
9	0.8	30.8639...	-3.73707...
10	0.9	30.4902...	-3.52129...
11	1.	30.1381...	-3.32509...

B11 = b10 + 0.1 * c10

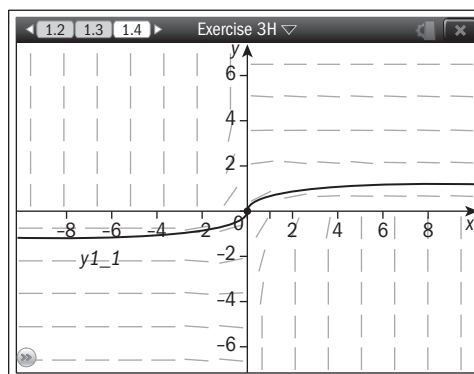
8 a



	xn	yn	f
1	0.	0.	1.
2	0.25	0.25	0.93941...
3	0.5	0.48485...	0.78472...
4	0.75	0.68103...	0.60003...
5	1.	0.83104...	0.43559...

B5 = b4 + 0.25 * c4

b



Review exercise

$$\begin{aligned} 1 \text{ a } \frac{dT}{dt} &= k(T-23) \Rightarrow \int \frac{dT}{T-23} = k \int dt \\ \ln|T-23| &= kt+c \Rightarrow |T-23| = e^{kt+c} \\ \therefore T &= Ae^{kt} + 23 \end{aligned}$$

$$b \text{ i } T(0) = 90 \Rightarrow A + 23 = 90 \Rightarrow A = 67$$

$$T(10) = 70 \Rightarrow 67e^{10k} + 23 = 70$$

$$\Rightarrow k = \frac{1}{10} \ln\left(\frac{47}{67}\right)$$

$$ii \text{ } T(t) = 40 \Rightarrow t = 38.7$$

$$2 \frac{dA}{dt} = -kA \Rightarrow \int \frac{dA}{A} = -k \int dt$$

$$\ln|A| = -kt + c \Rightarrow A = e^{-kt+c}$$

$$A(0) = A_0 \Rightarrow e^c = A_0$$

$$\therefore A = A_0 e^{-kt}$$

$$A = \frac{1}{2} A_0 \Rightarrow A_0 e^{-kt} = \frac{1}{2} A_0 \Rightarrow kt = \ln 2 \Rightarrow t = \frac{\ln 2}{k}$$

$$A_0 = 50 \text{ and } A(10) = 47 \Rightarrow k = -\frac{1}{10} \ln\left(\frac{47}{50}\right)$$

$$\therefore t = 112.02... = 112 \text{ to 3 s.f.}$$

$$3 \text{ } x \frac{dy}{dx} - y^2 = 1 \Rightarrow \frac{dy}{dx} = \frac{y^2+1}{x} \Rightarrow \int \frac{dy}{y^2+1} = \int \frac{dx}{x}$$

$$\therefore \arctan(y) = \ln|x| + c$$

$$y(3) = 0 \Rightarrow \arctan(0) = \ln(3) + c \Rightarrow c = -\ln(3)$$

$$\therefore \arctan(y) = \ln|x| - \ln(3) \Rightarrow y = \tan\left(\ln\left|\frac{x}{3}\right|\right)$$

$$4 \text{ a } 2x^2 \frac{dy}{dx} = x^2 + y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$

(Note that $f(\lambda x, \lambda y) = f(x, y)$)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = \frac{1+v^2}{2} \text{ QED}$$

$$b \text{ } v + x \frac{dv}{dx} = \frac{1+v^2}{2} \Rightarrow \frac{dv}{dx} = \frac{1}{x} \left(\frac{1+v^2}{2} - v \right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{x} \frac{(v-1)^2}{2} \Rightarrow \frac{2}{(v-1)^2} dv = \frac{dx}{x}$$

$\therefore$ the variables can be separated QED

$$\begin{aligned} c \int \frac{2}{(v-1)^2} dv &= \int \frac{dx}{x} \Rightarrow -\frac{2}{v-1} = \ln|x| + c \\ \Rightarrow v &= -\frac{2}{\ln|x| + c} - 1 \\ \therefore y &= -x \left(\frac{2}{\ln|x| + c} - 1 \right) \end{aligned}$$

$$5 \text{ } I(x) = e^{\int -2 \tan(x) dx} = \cos^2(x)$$

Multiplying both sides of $\frac{dy}{dx} - 2 \tan(x) \cdot y = 1$ by $I(x)$,

$\cos^2(x) \cdot \frac{dy}{dx} - 2 \cos(x) \sin(x) \cdot y = \cos^2(x)$ which is an exact DE

$$\therefore \cos^2(x) \cdot y = \int \cos^2(x) dx$$

As $\cos^2(x) = \frac{\cos(2x)+1}{2}$ it follows that

$$\cos^2(x) \cdot y = \int \frac{\cos(2x)+1}{2} dx$$

$$\Rightarrow \cos^2(x) \cdot y = \frac{1}{4} \sin(2x) + \frac{1}{2} x + c$$

$$y\left(\frac{\pi}{4}\right) = 2 \Rightarrow \frac{1}{4} + \frac{\pi}{8} + c = 2 \Rightarrow c = \frac{3}{4} - \frac{\pi}{8}$$

$$y = \frac{1}{2} \tan(x) + \left(\frac{1}{2} x + \frac{3}{4} - \frac{\pi}{8} \right) \sec^2(x)$$

$$6 \text{ a } \frac{dy}{dx} = \frac{3y^2+x^2}{2xy} \text{ (Note that } f(\lambda x, \lambda y) = f(x, y))$$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

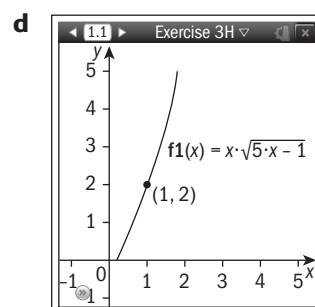
$v + x \frac{dv}{dx} = \frac{3v^2+1}{2v} \Rightarrow x \frac{dv}{dx} = \frac{v^2+1}{2v}$ which is variables separable DE

$$b \int \frac{2v}{v^2+1} dv = \int \frac{1}{x} dx \Rightarrow \ln(v^2+1) = \ln|x| + c$$

$$\therefore v^2+1 = A|x| \Rightarrow \left(\frac{y}{x}\right)^2 + 1 = A|x|$$

$$c \text{ } y(1) = 2 \Rightarrow A = 5$$

$$\left(\frac{y}{x}\right)^2 + 1 = 5x \Rightarrow y = x\sqrt{5x-1}, x \geq \frac{1}{5}$$



$$7 \quad (x^2 + y^2) + 2xy \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

(homogeneous DE)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$, substituting in the equation above,

$$v + x \frac{dv}{dx} = v + \frac{v^2 + 1}{2v} \Rightarrow x \frac{dv}{dx} = -\frac{3v^2 + 1}{2v} \text{ which is}$$

variables separable DE

$$\int \frac{2v}{3v^2 + 1} dv = -\int \frac{1}{x} dx \Rightarrow \frac{1}{3} \ln(3v^2 + 1) = -\ln|x| + c$$

$$x > 0 \Rightarrow 3v^2 + 1 = \frac{a}{x^3} \Rightarrow 3\left(\frac{y}{x}\right)^2 + 1 = \frac{a}{x^3}, a = e^c$$

$$\therefore x^3 + 3xy^2 = a, a > 0 \text{ QED}$$

$$8 \text{ a} \quad \frac{dy}{dx} = y \tan(x) + 1 \Rightarrow \cos(x) \cdot \frac{dy}{dx}$$

$$= \cos(x) \cdot (y \tan(x) + 1)$$

$$\Rightarrow \cos(x) \frac{dy}{dx} - \sin(x) \cdot y = \cos(x) \text{ (exact DE)}$$

$\therefore I(x) = \cos(x)$ is an integrating factor.

$$b \quad \cos(x) \frac{dy}{dx} - \sin(x) \cdot y = \cos(x)$$

$$\Rightarrow \cos(x) \cdot y = \int \cos(x) dx$$

$$\cos(x) \cdot y = \sin(x) + c$$

$$\therefore y = \tan(x) + \frac{c}{\cos(x)}$$

$$y(0) = 2 \Rightarrow c = 2$$

$$\therefore y = \tan(x) + \frac{2}{\cos(x)} \text{ (or } \therefore y = \tan(x) + 2 \sec(x))$$

9 a

	A	B	C	D	E
	n	xn	yn	f	
			=0+0.2*n		
1	0	0.		1	2.
2	1	0.2		1.4	1.92929
3	2	0.4		1.78586	1.81397
4	3	0.6		2.14865	1.64583
5					
C4	=C3+D3*0.2				

Answer: 2.149

$$b \quad \frac{dy}{dx} + \frac{xy}{4-x^2} = 2 \text{ is a linear DE}$$

$$I(x) = e^{\int \frac{x}{4-x^2} dx} = e^{-\frac{1}{2} \ln|4-x^2|} = \frac{1}{\sqrt{4-x^2}}$$

Multiplying both sides of $\frac{dy}{dx} + \frac{xy}{4-x^2} = 2$ by $I(x)$,

$$\frac{1}{\sqrt{4-x^2}} \left(\frac{dy}{dx} + \frac{xy}{4-x^2} \right) = \frac{2}{\sqrt{4-x^2}}$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{\sqrt{4-x^2}} \cdot y \right) = \frac{2}{\sqrt{4-x^2}}$$

which is an exact DE

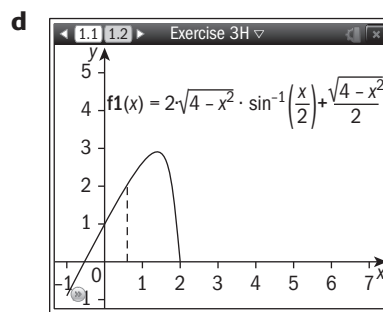
$$\therefore \frac{1}{\sqrt{4-x^2}} \cdot y = \int \frac{2}{\sqrt{4-x^2}} dx$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}} \cdot y = 2 \arcsin\left(\frac{x}{2}\right) + c$$

$$y(0) = 1 \Rightarrow c = \frac{1}{2}$$

$$\therefore y = 2\sqrt{4-x^2} \arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\sqrt{4-x^2}$$

$$c \quad y(0) = 2.117 \text{ (to 3 d.p.) (use GDC)}$$



The graph is increasing for $0 < x < 0.6$ which explains why the approximate solution is greater than the exact one.

$$10 \text{ a} \quad \frac{dV}{dt} = -\alpha V \Rightarrow \int \frac{dV}{V} = -\alpha \int dt$$

$$\Rightarrow \ln V = -\alpha t + c$$

$$\therefore V = Ae^{-\alpha t}$$

$$V(0) = V_0 \Rightarrow A = V_0$$

$$\therefore V = V_0 e^{-\alpha t}$$

$$b \quad V = \frac{V_0}{3} \Rightarrow V_0 e^{-\alpha t} = \frac{V_0}{3} \Rightarrow -\alpha t = \ln\left(\frac{1}{3}\right)$$

$$\therefore t = \frac{\ln 3}{\alpha}$$

$$11 \quad xy \frac{dy}{dx} = 1 + y^2 \text{ is a variable separable DE}$$

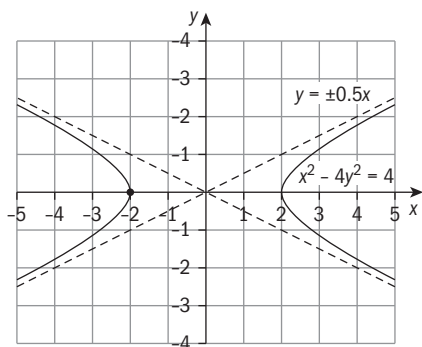
$$\int \frac{y}{1+y^2} dy = \int \frac{1}{x} dx \Rightarrow \frac{1}{2} \ln(y^2 + 1) = \ln|x| + c$$

$$y(2) = 0 \Rightarrow c = \ln\left(\frac{1}{2}\right)$$

$$\frac{1}{2} \ln(y^2 + 1) = \ln|x| + \ln\left(\frac{1}{2}\right) \Rightarrow \sqrt{y^2 + 1} = \frac{x}{2}$$

$$\therefore x^2 - 4y^2 = 4 \text{ or}$$

$$y = \pm \frac{\sqrt{x^2 - 4}}{2}, D =]-\infty, -2[\cup]2, +\infty[$$



$$12 \quad kx = mv \frac{dv}{dx} \Rightarrow k \int x dx = m \int v dv$$

$$k \frac{x^2}{2} = m \frac{v^2}{2} + c$$

$$v(0) = v_0 \Rightarrow m \frac{v_0^2}{2} + c = 0 \Rightarrow c = -m \frac{v_0^2}{2}$$

$$k \frac{x^2}{2} = m \frac{v^2}{2} - m \frac{v_0^2}{2} \Rightarrow kx^2 = mv^2 - mv_0^2$$

$$v = \pm \sqrt{\frac{kx^2 + mv_0^2}{m}}$$

$$\therefore v(1) = \pm \sqrt{\frac{k + mv_0^2}{m}}$$

$$13 \quad \frac{1}{4x - x^2} = \frac{a}{x} + \frac{b}{x - 4} \Rightarrow \frac{1}{4x - x^2} = \frac{(a + b)x - 4a}{4x - x^2}$$

$$\Rightarrow a = -b \text{ and } 4b = 1$$

$$\therefore a = -\frac{1}{4}, b = \frac{1}{4}$$

$$\frac{dx}{dt} = kx(4 - x) \Rightarrow \frac{dx}{4x - x^2} = k dt$$

(variable separable DE)

$$\Rightarrow \frac{1}{4} \int \left(-\frac{1}{x} - \frac{1}{4 - x} \right) dx = k \int dt$$

$$-\frac{1}{4} (\ln x + \ln(4 - x)) = kt + c$$

$$\therefore 4x - x^2 = Ae^{-4kt}, 0 < x < 4$$

$$14 \text{ a } \frac{d^2 y}{dx^2} = \frac{1}{125l^3} (x^2 - lx)$$

$$z = \frac{1}{125l^3} \left(\frac{x^3}{3} - \frac{lx^2}{2} \right) + \frac{1}{1500}$$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{125l^3} \left(\frac{3 \cdot x^2}{3} - \frac{2lx}{2} \right)$$

$$\therefore \frac{dz}{dx} = \frac{1}{125l^3} (x^2 - lx) \text{ QED}$$

$$\text{b } \frac{dy}{dx} = z \Rightarrow \frac{dy}{dx} = \frac{1}{125l^3} \left(\frac{x^3}{3} - \frac{lx^2}{2} \right) + \frac{1}{1500}$$

$$y = \frac{1}{125l^3} \left(\frac{x^4}{12} - \frac{lx^3}{6} \right) + \frac{x}{1500} + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$\therefore y = \frac{1}{125l^3} \left(\frac{x^4}{12} - \frac{lx^3}{6} \right) + \frac{x}{1500} \text{ or}$$

$$\therefore y = \frac{1}{125l^3} \left(\frac{x^4}{12} - \frac{lx^3}{6} + \frac{l^3 x}{12} \right)$$

$$\text{c } \frac{dy}{dx} = \frac{1}{125l^3} \left(\frac{x^3}{3} - \frac{lx^2}{2} + \frac{l^3}{12} \right) \Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{125l^3} (x^2 - lx)$$

$$\text{d } y(l) = \frac{1}{125l^3} \left(\frac{l^4}{12} - \frac{l^4}{6} + \frac{l^4}{12} \right) = 0$$

$$\text{e } y(2) = \frac{1}{125 \times 4^3} \left(\frac{2^4}{12} - \frac{4 \times 2^3}{6} + \frac{4^3 \times 2}{12} \right) = \frac{1}{1200};$$

maximum distance below [AB]

4

The finite in the infinite

Exercise 4A

- 1 a $\lim_{n \rightarrow \infty} 3^n \neq 0$, hence the series diverges.
- b $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with $r = \frac{1}{3}$, hence it converges to its sum, $S_{\infty} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} = 1.5$.
- c $\lim_{n \rightarrow \infty} \frac{2n!}{3n! + 3} = \lim_{n \rightarrow \infty} \frac{2}{3 + \frac{3}{n!}} = \frac{2}{3} \neq 0$, hence the series diverges.
- d $\sum_{n=0}^{\infty} 3 \cdot \left(\frac{1}{7}\right)^{n-1} = 3 \cdot 7 + \sum_{n=1}^{\infty} 3 \cdot \left(\frac{1}{7}\right)^{n-1} = 3 \cdot \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^{n-1}$ is a geometric series with $r = \frac{1}{7}$, hence converges to its sum, $S_{\infty} = 3 \cdot \frac{1}{1 - \frac{1}{7}} = \frac{21}{6} = 3\frac{1}{2}$. Hence $S = 24.5$.
- e $\sum_{n=1}^{\infty} \frac{e^{n\pi}}{\pi^{n^e}} = \sum_{n=1}^{\infty} \left(\frac{e^{\pi}}{\pi^e}\right)^n$ is a geometric series with $r = \frac{e^{\pi}}{\pi^e} \approx 1.03$. Since $r > 1$ the series does not converge.
- f $\frac{3}{2} + \frac{6}{3} + \frac{9}{4} + \frac{12}{5} \dots = \sum_{n=1}^{\infty} \frac{3n}{n+1} \cdot \lim_{n \rightarrow \infty} \frac{3n}{n+1}$

$$= \lim_{n \rightarrow \infty} \frac{3}{1 + \frac{1}{n}}$$

$$= 3 \neq 0$$
, hence series diverges.
- g $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{6}{2^n} = 6 \sum_{n=1}^{\infty} \frac{1}{2^n} = 6 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}$ is an infinite geometric series, $r = \frac{1}{2}$, hence $S_{\infty} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \Rightarrow S_{\infty} = 6$.
- h $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$. Both series are geometric with $r = \frac{1}{2}$ and $r = \frac{1}{3}$ respectively. Hence $S_{\infty} = \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{3}} = 2 - \frac{3}{2} = 0.5$.

$$\text{i } \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1} + 1} = \lim_{n \rightarrow \infty} \frac{2^n}{2 \cdot 2^n + 1} = \lim_{n \rightarrow \infty} \frac{1}{2 + \frac{1}{2^n}} = \frac{1}{2} \neq 0,$$

hence the series diverges by the nth term test.

$$\begin{aligned} 2 \quad \frac{1}{2(n-1)} - \frac{1}{2(n+1)} &= \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ &= \frac{1}{2} \left(\frac{(n+1) - (n-1)}{n^2 - 1} \right) = \frac{1}{2} \left(\frac{2}{n^2 - 1} \right) = \frac{1}{n^2 - 1}. \end{aligned}$$

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) &= \frac{1}{2} \left(\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \right. \\ &\quad \left. + \left(\frac{1}{6} - \frac{1}{8}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \dots \right) \\ &= \frac{1}{2} \left(1 + \frac{1}{2} + \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+2} \right) \right) = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} 3 \quad 3 + \frac{9}{x} + \frac{27}{x^2} + \dots &= \sum_{n=1}^{\infty} \frac{3^n}{x^{n-1}}. \text{ This is a geometric series with } r = \frac{3}{x}. \text{ For the series to converge,} \\ \left| \frac{3}{x} \right| < 1 &\Rightarrow x < -3, x > 3. S_{\infty} = \frac{3}{1 - \frac{3}{x}} = \frac{3x}{x-3}. \end{aligned}$$

Exercise 4B

$$1. \text{ Corollary 1: } \int_x^a f(t) dt = - \int_a^x f(t) dt$$

By the FTC, $g(x) = \int_x^a f(t) dt$, $b \leq x \leq a$ is an anti-derivative of f , i.e., $g'(x) = f(x)$, $b < x < a$

$$= - \int_x^a f(t) dt.$$

$$\text{Corollary 2: } \int_a^b f(x) dx = F(b) - F(a)$$

Let $g(x) = \int_a^x f(t) dt$, $a \leq x \leq b$. If furthermore F is

any anti-derivative of f , then by the mean value theorem, $F(x) = g(x) + c$, for some constant c .

Evaluating therefore $F(b) - F(a)$,

$$\begin{aligned} F(b) - F(a) &= [g(b) + c] - [g(a) + c] \\ &= g(b) - g(a) \\ &= \int_a^b f(t) dt - \int_a^a f(t) dt \\ &= \int_a^b f(t) dt - 0 \\ &= \int_a^b f(t) dt \end{aligned}$$

$$\begin{aligned}
 2 \quad \sum_{i=1}^n f(c_i \Delta x_i) &= \sum_{i=1}^n \sin(c_i) \frac{\pi}{n} = \sum_{i=1}^n \sin\left(\frac{i-1}{n}\right) \frac{\pi}{n} \\
 &= \frac{\pi}{n} \left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) \right. \\
 &\quad \left. + \dots + \sin\left(\frac{(n-1)\pi}{n}\right) \right)
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \pi \lim_{n \rightarrow \infty} \frac{\pi}{n} &\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) \right. \\
 &\quad \left. + \sin\left(\frac{3\pi}{n}\right) + \dots + \sin\left(\frac{(n-1)\pi}{n}\right) \right) \\
 &= \int_0^\pi \sin x \, dx = -\cos x \Big|_0^\pi = 2
 \end{aligned}$$

And therefore,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{\pi}{n} &\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) \right. \\
 &\quad \left. + \sin\left(\frac{3\pi}{n}\right) + \dots + \sin\left(\frac{(n-1)\pi}{n}\right) \right) \\
 &= \frac{2}{\pi}
 \end{aligned}$$

$$3 \quad i \quad \Delta x = \frac{5-0}{n} = \frac{5}{n}; \quad x_i = 0 + i \frac{5}{n} = \frac{5i}{n};$$

$$f(x_i) = 2 \cdot \frac{5i}{n} = \frac{10i}{n}$$

$$\begin{aligned}
 f(x_i) \Delta x &= \sum_{i=1}^n \left(\frac{10i}{n} \right) \left(\frac{5}{n} \right) = \sum_{i=1}^n \frac{50i}{n^2} = \frac{50}{n^2} \sum_{i=1}^n i \\
 &= \frac{50}{n^2} \left(\frac{n(n+1)}{2} \right) = 25 \cdot \frac{n+1}{n}
 \end{aligned}$$

$$\int_0^5 2x \, dx = \lim_{n \rightarrow \infty} \left(25 \cdot \frac{n+1}{n} \right) = 25 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 25$$

$$ii \quad \Delta x = \frac{0 - (-2)}{n} = \frac{2}{n};$$

$$\begin{aligned}
 x_i &= -2 + i \frac{2}{n}; \quad f(x_i) = 3 \left(-2 + \frac{2i}{n} \right)^2 + 2 \left(-2 + \frac{2i}{n} \right) \\
 &= \left(12 - \frac{24i}{n} + \frac{12i^2}{n^2} \right) - 4 + \frac{4i}{n} = 8 - \frac{20i}{n} + \frac{12i^2}{n^2}
 \end{aligned}$$

$$f(x_i) \Delta x = \sum_{i=1}^n \left(\frac{16}{n} - \frac{40i}{n^2} + \frac{24i^2}{n^3} \right)$$

$$= \frac{16}{n} - \frac{40}{n^2} \sum_{i=1}^n i + \frac{24}{n^3} \sum_{i=1}^n i^2$$

$$= \frac{16}{n} - \frac{40}{n^2} \left(\frac{n(n+1)}{2} \right) + \frac{24}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right)$$

$$\int_{-2}^0 (3x^2 + 2x)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{16}{n} - \frac{40}{n^2} \left(\frac{n(n+1)}{2} \right) + \frac{24}{n^3} \left(\frac{n(n+1)(2n+1)}{6} \right) \right) = 4$$

$$4 \quad a \quad f'(x) = \sqrt{3+4x}$$

$$b \quad \int_x^4 2 \tan t \, dt = -\int_4^x 2 \tan t \, dt = -2 \tan x$$

$$c \quad \int_t^0 t \sec t \, dt = -\int_0^t t \sec t \, dt = -x \sec x$$

$$5 \quad a \quad F(4) = \int_4^4 \sqrt{t^2 + 9t} \, dt = 0$$

$$b \quad F'(x) = \sqrt{x^2 + 9}; \quad F'(4) = \sqrt{25} = 5$$

$$c \quad F''(x) = \frac{x}{\sqrt{x^2 + 9}}; \quad F''(4) = \frac{4}{5}$$

$$6 \quad G'(x) = \sin x$$

7 The curve will be concave up at $f''(x) > 0$.

$$y' = \frac{1}{1+x+x^2}$$

$$y''(x) = \frac{-(1+2x)}{(1+x+x^2)^2}$$

$$y'' > 0 \Rightarrow -(1+2x) > 0, \quad x < -\frac{1}{2}$$

Exercise 4C

$$\begin{aligned}
 1 \quad \int_0^\infty e^{-x} \, dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} \, dx = \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{e^b} + 1 \right) = 1,
 \end{aligned}$$

hence the integral converges to 1.

$$\begin{aligned}
 2 \quad \int_0^\infty \sin x \, dx &= \lim_{b \rightarrow \infty} \int_0^b \sin x \, dx = \lim_{b \rightarrow \infty} (-\cos x) \Big|_0^b \\
 &= \lim_{b \rightarrow \infty} (-\cos b + 1). \text{ Since } \cos b \text{ oscillates} \\
 &\text{between } -1 \text{ and } 1, \text{ the integral diverges.}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \int_0^\infty \frac{e^x}{1+e^{2x}} \, dx &= \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1+e^{2x}} \, dx = \lim_{b \rightarrow \infty} \left[\arctan e^x \right]_0^b = \frac{\pi}{4}, \\
 &\text{hence the integral converges to } \frac{\pi}{4}.
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \int_1^\infty \frac{1}{\sqrt{x}} \, dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x}} \, dx = \lim_{b \rightarrow \infty} (2\sqrt{x}) \Big|_1^b = \infty, \\
 &\text{hence the integral diverges.}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \int_0^\infty \cos \pi x \, dx &= \lim_{b \rightarrow \infty} \int_0^b \cos \pi x \, dx \\
 &= \pi \lim_{b \rightarrow \infty} \sin x \Big|_0^b = \pi \lim_{b \rightarrow \infty} (\sin b - 1). \text{ Since } \sin b \\
 &\text{oscillates between } -1 \text{ and } 1, \text{ the integral diverges.}
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \int_0^\infty x^2 e^{-x} \, dx &= \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} \, dx. \text{ Using integration by} \\
 &\text{parts, } \lim_{b \rightarrow \infty} \int_0^b x^2 e^{-x} \, dx = \lim_{b \rightarrow \infty} \left[-e^{-x} (x^2 - 2x - 2) \right]_0^b = 2, \\
 &\text{hence the integral converges.}
 \end{aligned}$$

Exercise 4D

- 1 From the GDC, $f(x) = \frac{3}{x+1}$ is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.
- $$\int_1^{\infty} \frac{3}{x+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{3}{x+1} dx = 3 \lim_{b \rightarrow \infty} [\ln(x+1)]_1^b$$
- $$= 3 \lim_{b \rightarrow \infty} (\ln(b+1) - \ln 2) = 3 \lim_{b \rightarrow \infty} \ln \frac{b+1}{2} = \infty$$
- hence the series diverges.
- 2 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\frac{\ln(x^2+1)}{2} \right]_1^b$$

$$= \frac{1}{2} \lim_{b \rightarrow \infty} [\ln(b^2+1) - \ln 2] = \infty,$$

hence the series diverges.

- 3 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \arctan x \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} [\arctan b - \arctan 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

hence the series converges.

- 4 The integral test cannot be applied since the function is increasing for $x \geq 1$. However, since $\lim_{b \rightarrow \infty} b \sin\left(\frac{1}{b}\right) = 1 \neq 0$, the series diverges by the n th test for divergence.

- 5 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{e^x}{1+e^{2x}} dx = \lim_{b \rightarrow \infty} [\arctan e^x]_0^b = \frac{\pi}{2}$$

Hence, the series converges.

- 6 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.
- 7 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{4x}{(2x^2+3)^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{4x}{(2x^2+3)^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2x^2+3} \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{2b^2+3} + \frac{1}{5} \right) = \frac{1}{5}$$

Hence, the series converges.

- 8 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.

- 9 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.
- 10 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \ln(\ln(x)) \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} (\ln(\ln(b)) - \ln(\ln(1))) = \infty$$

Hence the series diverges.

- 11 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{\arctan x}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\arctan x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\frac{(\arctan x)^2}{2} \right]_1^b = \frac{3\pi^2}{32}$$

Hence the series converges.

- 12 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{\ln x + 1}{x} \right]_1^b$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{\ln b + 1}{b} + \frac{\ln 2}{2} \right) = \frac{\ln 2}{2}$$

Hence the series converges.

- 13 From the GDC, the function is continuous, positive and decreasing for all $x \geq 2$, hence we can apply the integral test.

$$\int_1^{\infty} \frac{x^2}{e^x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x^2}{e^x} dx = \lim_{b \rightarrow \infty} (-e^{-x}(x^2 + 2x + 2)) \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} \left((-e^{-b}(b^2 + 2b + 2)) + \frac{5}{e} \right) = \frac{5}{e}.$$

Hence the series converges.

Exercise 4E

- 1 Using the p -series test, $p = \pi$, $\pi > 1$, hence the series converges.
- 2 Using the p -series test, $p = \frac{\pi}{4}$, $\frac{\pi}{4} < 1$, hence the series diverges.
- 3 $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$
Using the p -series test, $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, $p = \frac{3}{2}$, $\frac{3}{2} > 1$, hence the series converges.
- 4 $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$
Using the p -series test, $p = \frac{2}{3}$, $\frac{2}{3} < 1$ hence the series diverges.
- 5 $p = \frac{1}{2}$, $\frac{1}{2} < 1$, hence the series diverges.

Exercise 4F

- $\frac{1}{n^3+1} < \frac{1}{n^3}$, $n \geq 1$. Since $\frac{1}{n^3}$ converges by the p-series test, the original series also converges.
- $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n-1}}$, $n \geq 2$. Since $\frac{1}{\sqrt{n}}$ diverges by the p-series test, the original series also diverges.
- $\frac{1}{(n+1)!} < \frac{1}{n^2}$, $n \geq 1$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.
- $\ln(n) \leq n$ for $n \geq 2$, hence $\frac{1}{\ln n} \geq \frac{1}{n}$, $n \geq 2$. Since $\frac{1}{n}$ diverges by the p-series test, the original series diverges.
- $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$, $n \geq 1$. Since $\frac{1}{\sqrt{n^3}}$ converges by the p-series test, the original series converges.
- $\ln(n) > 1$ for $n \geq 3$, hence $\frac{\ln n}{n} \geq \frac{1}{n}$ for $n \geq 3$. Since $\frac{1}{n}$ diverges by the p-series test, the original series diverges.
- $\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$ for $n \geq 1$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.
- $\frac{n-1}{n^2\sqrt{n}} = \frac{n-1}{n^{\frac{5}{2}}}$; $\frac{n-1}{n^{\frac{5}{2}}} < \frac{1}{n^{\frac{5}{2}}}$ for $n \geq 1$. Since $\frac{1}{n^{\frac{5}{2}}}$ converges by the p-series test, the original series converges.
- $\arctan x < \frac{\pi}{2}$; $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$ for $n \geq 1$. Hence $\sum \arctan \frac{n}{\sqrt{n^3+1}} < \frac{\pi}{2} \sum \frac{1}{\sqrt{n^3}}$. Since $\frac{1}{\sqrt{n^3}}$ converges by the p-series test, the original series also converges.
- $\ln(n) \geq 2$ for $n \geq e^2$ hence $\frac{1}{n^{\ln n}} \leq \frac{1}{n^2}$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.

Exercise 4G

- Compare with $\sum \frac{1}{\sqrt{n^2}}$. Hence,
$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{\sqrt{n^2}}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^2}{n^2+1}} = \lim_{n \rightarrow \infty} \sqrt{\frac{1}{1+\frac{1}{n^2}}} = 1$$
, and the series diverges since $\sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$ diverges.

- Compare with $\sum \frac{n^2}{n^5}$. Hence,
$$\lim_{n \rightarrow \infty} \frac{\frac{2n^2-1}{4n^5+n-1}}{\frac{n^2}{n^5}} = \lim_{n \rightarrow \infty} \frac{2n^7-2n^2}{4n^7+n^6-n^5} = \frac{1}{2}$$
. The series converges since by the p-series test, $\sum \frac{n^2}{n^5} = \sum \frac{1}{n^3}$ converges.
- Compare with $\sum \frac{1}{n}$. Hence $\lim_{n \rightarrow \infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = 1$. The series diverges, since $\sum \frac{1}{n}$ diverges.
- Compare with $\sum \frac{1}{2^n}$, a convergent geometric series. Since $\lim_{n \rightarrow \infty} \frac{\frac{1}{2^{n-1}}}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n-1} = 1$, the series converges.
- Compare with $\sum \frac{n^2}{\sqrt{n^5}}$.
$$\lim_{n \rightarrow \infty} \frac{\frac{3n^2+2n}{\sqrt{4+n^5}}}{\frac{n^2}{\sqrt{n^5}}} = \lim_{n \rightarrow \infty} \left(\frac{3n^2+2n}{n^2} \times \sqrt{\frac{n^5}{4+n^5}} \right) = 3$$
. Since $\sum \frac{n^2}{\sqrt{n^5}} = \sum \frac{1}{\sqrt{n}}$, the series diverges.
- $\sum \left(\frac{1}{2^n-1} - \frac{1}{2^n} \right) = \sum \frac{1}{4n^2-2n}$. Compare with $\sum \frac{1}{n^2}$.
$$\lim_{n \rightarrow \infty} \frac{\frac{1}{4n^2-2n}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{4n^2-2n} = \frac{1}{4}$$
. Hence, since $\sum \frac{1}{n^2}$ converges by the p-series test, the series converges.
- Compare with $\sum \frac{1}{\sqrt{n}}$.
$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2\sqrt{n+2n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt{n+2}} = \frac{1}{3}$$
. Hence, since $\sum \frac{1}{\sqrt{n}}$ diverges by the p-series test, the series diverges.
- Compare with $\sum \frac{n2^n}{n^2} = \sum \frac{2^n}{n}$, which diverges by the nth term test for divergence, since $\lim_{n \rightarrow \infty} \frac{2^n}{n^2} \neq 0$.
$$\lim_{n \rightarrow \infty} \frac{\frac{n2^{n+1}}{3n^2+2n}}{\frac{2^n}{n}} = \lim_{n \rightarrow \infty} \frac{n^2 2^n + n}{3n^2 2^n + 2n 2^n} = \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{n2^n}}{3 + \frac{2}{n}} = \frac{1}{3}$$
. Hence the series diverges.

Exercise 4H

- $\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \cdot \frac{2^{n+1}}{2^n} \right) = 2 \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{n}} = 2$. Since $L > 1$ the series diverges.
- $\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$. Since $L < 1$ the series converges.
- $\lim_{n \rightarrow \infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n \rightarrow \infty} \left(\frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \right) = 3 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$. Since $L < 1$ the series converges.

$$4 \quad \lim_{n \rightarrow \infty} \frac{\frac{3}{(n+1)2^{n+1}}}{\frac{3}{n2^n}} = \lim_{n \rightarrow \infty} \frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n}{n+1} = \frac{1}{2}.$$

Since $L < 1$ the series converges.

$$5 \quad \lim_{n \rightarrow \infty} \frac{\frac{\pi^{n+1}}{3(n+1)+2}}{\frac{\pi^n}{3n+2}} = \pi \lim_{n \rightarrow \infty} \frac{3n+2}{3n+5} = \pi.$$

Since $L > 1$ the series diverges.

$$6 \quad \lim_{n \rightarrow \infty} \frac{\frac{5^{n+3}}{(n+2)!}}{\frac{5^{n+2}}{(n+1)!}} = 5 \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+2)!} = 5 \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0.$$

Since $L < 1$ the series converges.

Exercise 4I

- 1 Using the ratio test,

$$\lim_{n \rightarrow \infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n \rightarrow \infty} \left(\frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} \right) = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{1}{2}.$$

Since $L < 1$ the series converges by the ratio test.
Since the series converges absolutely, the series converges.

- 2 $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ has terms of different signs, since
 $-1 \leq \sin n \leq 1$. Since $\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$ for all n , and $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges by the p-series test, the series converges absolutely, therefore the series converges.

- 3 Using the ratio test,

$$\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = 2 \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = 2 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Since $L < 1$ the series converges by the ratio test.
Since the series converges absolutely, the series converges.

- 4 Using the comparison test, $\lim_{n \rightarrow \infty} \frac{\arctan n}{n^3} \leq \frac{\frac{\pi}{2}}{n^3}$
 for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-series test, so does $\sum_{n=1}^{\infty} \frac{\arctan n}{n^3}$. Hence the series converges absolutely.

- 5 Using the ratio test,

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = \lim_{n \rightarrow \infty} \frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Since $L < 1$, the series converges. Hence the series converges absolutely.

- 6 Since $\lim_{n \rightarrow \infty} \frac{n!}{e^n} \neq 0$, the series does not converge absolutely.

Exercise 4J

- 1 a The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges.
 For the series of absolute values, $\ln(n) \leq n$ for $n \geq 2$, hence $\frac{1}{\ln n} \geq \frac{1}{n}$, $n \geq 2$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test, the series of absolute values diverges by the comparison test. Hence the series converges conditionally.

- b The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges.
 Using the ratio test for the series of absolute values,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} &= \lim_{n \rightarrow \infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n \rightarrow \infty} \left[\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \cdot \frac{1}{2} \right] = \frac{1}{2}. \end{aligned}$$

Since $L < 1$, the series converges absolutely.

- c The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges. The series of absolute values diverges by the p-series test, hence the series converges conditionally.

- d The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges. For the series of absolute values, using the ratio test,

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Since $L < 1$, the series converges absolutely.

- e The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges. The series of absolute values converges because it is a geometric series, with $|r| < 1$. Hence the series converges absolutely.

- f $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots + (-1)^n \frac{1}{n} + \dots$

The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges. The series of absolute values is the harmonic series, which diverges. Hence the series converges conditionally.

- g The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges.

The integral test can be used on the series of absolute values, since $f(x) = \frac{1}{x \ln x}$ is continuous, positive and decreasing for all $x \geq 1$.

$$\begin{aligned} \text{Hence, } \int_1^{\infty} \frac{1}{x \ln x} &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x \ln x} = \lim_{n \rightarrow \infty} [\ln(\ln x)]_1^b \\ &= \lim_{n \rightarrow \infty} [\ln(\ln b) - \ln(\ln 1)] = \infty. \end{aligned}$$

Since the series of absolute values diverges, the series converges conditionally.

- h The absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges. The series of absolute values converges by comparison with $\sum_{n=2}^{\infty} \frac{\pi}{n^2}$, hence the series converges absolutely.

- 2 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$. Hence the truncation error is

$$u_{11} = \frac{1}{11^3} = \frac{1}{1331}. S = S_{10} + \frac{1}{1331} \approx 0.902.$$

- 3 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges.

$$u_{n+1} \leq 0.001 \Rightarrow \frac{1}{(n+1)^4} \leq 0.001 \Rightarrow n = 5. \text{ Hence, } S_5 \approx 0.948.$$

- 4 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, hence the series converges.

Since the series begins with $n = 2$, and we want the first three terms, i.e., $n = 2, 3, 4$, therefore

$$u_5 = \frac{1}{5^3 \times 3^5} = \frac{1}{30375}. S = S_3 + u_5 = -0.012644.$$

- 5 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$.

$$u_{n+1} \leq 10^{-5} \Rightarrow \frac{1}{(n+1)^3 \ln(n+1)} < 10^{-5} \Rightarrow n = 30.$$

Hence, the truncated series has the required accuracy after the 29th term, since the first term corresponds to $n = 2$, and not $n = 1$.

Exercise 4K

- 1 Using the ratio test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{6^{n+1}}{(n+1)!}}{\frac{6^n}{n!}} &= \lim_{n \rightarrow \infty} \left(\frac{6^{n+1}}{6^n} \cdot \frac{n!}{(n+1)!} \right) \\ &= 6 \lim_{n \rightarrow \infty} \frac{1}{n+1} = 6 \cdot 0 = 0. \end{aligned}$$

Since $L < 1$, the series converges.

- 2 Using the ratio test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(n+1)2^{n+1}}{3^{n+1}}}{\frac{n2^n}{3^n}} &= \lim_{n \rightarrow \infty} \left(\frac{2 \cdot 2^n (n+1)}{3 \cdot 3^n} \cdot \frac{3^n}{n2^n} \right) = \frac{2}{3} \lim_{n \rightarrow \infty} \frac{n+1}{n} \\ &= \frac{2}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = \frac{2}{3}. \end{aligned}$$

Since $L < 1$, the series converges.

- 3 This is a geometric series whose common ratio is $\frac{1}{e}$. Since $|r| < 1$, the series converges to its sum.

- 4 Using the GDC, we see that the function that represents the series is continuous, positive and decreasing for all $n \geq 1$. Hence, we can use the integral test. Using the substitution $u = \frac{1}{x}$,

$$\begin{aligned} \int_1^\infty \frac{1}{x^2} \tan\left(\frac{1}{x}\right) dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} \tan\left(\frac{1}{x}\right) dx \\ &= \lim_{n \rightarrow \infty} \ln\left(\frac{1}{\cos x}\right)_b^\infty = -\ln(\cos(1)). \end{aligned}$$

Hence the series converges.

- 5 Using the limit comparison test with $\sum \frac{n}{n^2}$, or $\sum \frac{1}{n}$,

$$\lim_{n \rightarrow \infty} \frac{\frac{2n+1}{2n^2+3n-1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{2n^2+3n-1} = 1. \text{ Since } L \text{ is a}$$

positive real number, the series diverges since the harmonic series diverges.

- 6 Since the absolute value of the terms of the series is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, the series converges.

For all $n \geq 1$, $\sqrt{3n} > \sqrt{3n-1}$, hence $\frac{1}{\sqrt{3n}} < \frac{1}{\sqrt{3n-1}}$.

Since $\sum \frac{1}{\sqrt{n}}$ diverges by the p-series test, so does

$\frac{1}{\sqrt{3}} \sum \frac{1}{\sqrt{n}}$. Hence the series does not converge absolutely, and converges conditionally.

- 7 $\sum \frac{2}{\ln n^2} = \sum \frac{2}{2 \ln n} = \sum \frac{1}{\ln n}$. Since $\frac{1}{n} < \frac{1}{\ln n}$ for all $n \geq 2$, the series diverges by comparison with the harmonic series.

- 8 Since $\lim_{n \rightarrow \infty} \frac{n(n+1)}{\sqrt{n^3+5n^2}} \neq 0$, the series diverges by the Divergence Test.

- 9 Since the absolute value of the terms of the series is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$, the series converges.

Using the integral test, since the function representing the series is positive, decreasing and continuous for all $x \geq 2$,

$$\begin{aligned} \int_2^\infty \frac{1}{x\sqrt{\ln x}} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x\sqrt{\ln x}} dx \\ &= \lim_{b \rightarrow \infty} \left[2\sqrt{\ln x} \right]_2^b \\ &= \lim_{b \rightarrow \infty} (2\sqrt{\ln b} - 2\sqrt{\ln 2}) = \infty. \end{aligned}$$

Hence, the series diverges absolutely, and converges conditionally.

- 10 Using the ratio test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{4^{n+1}(n+1)^2}{(n+1)!}}{\frac{4^n n^2}{n!}} &= \lim_{n \rightarrow \infty} \left(\frac{4^{n+1}(n+1)^2}{(n+1)!} \times \frac{n!}{4^n n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{4(n+1)}{n^2} = 0. \end{aligned}$$

Since $L < 1$, the series converges.

11 Using the ratio test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3(n+1)+2)}}{\frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}} \\ = \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{n!} \cdot \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{2 \cdot 5 \cdot 8 \cdots (3(n+1)+2)} \right) \\ = \lim_{n \rightarrow \infty} \frac{(n+1)}{3n+5} = \frac{1}{3} \end{aligned}$$

Since $L < 1$ the series converges.

12 $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \frac{4^n}{n^n}$. Using the ratio test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{4^{n+1}}{(n+1)^{n+1}}}{\frac{4^n}{n^n}} &= \lim_{n \rightarrow \infty} \left(\frac{4 \cdot 4^n}{(n+1)(n+1)^n} \times \frac{n^n}{4^n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{4n^n}{(n+1)(n+1)^n} = 0. \end{aligned}$$

Hence, the series diverges. (This one is best done with the Root Test, which is not on the syllabus.)

13 Using the limit comparison test with $\sum \frac{1}{n\sqrt{n}}$,

$$\lim_{n \rightarrow \infty} \frac{\frac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}}{\frac{1}{n\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1, \text{ and } 1 > 0, \text{ hence the}$$

series converges since $\sum \frac{1}{n\sqrt{n}}$ converges by the p-series test.

14 $0 < \frac{\arctan n}{\frac{\pi}{2}} < \frac{\pi}{2} \cdot \sum \frac{\pi}{2} = \frac{\pi}{2} \sum \frac{1}{n^{\frac{3}{2}}}$. Since this

series converges by the p-series test, the original series also converges.

15 a The series is continuous and decreasing, but not positive for all $n \geq 1$.

b The series does not have all non-negative terms.

c The series does not have all non-negative terms.

d i the sequence of absolute values is not decreasing.

ii the limit of the n th term as n goes to infinity does not equal 0.

16 a $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

$$S_1 = \frac{1}{2!}; S_2 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}; S_3 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{23}{24};$$

$$S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{119}{120}$$

$$S_5 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} = \frac{719}{720};$$

$$S_6 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \frac{6}{7!} = \frac{5039}{5040}.$$

$$\text{Conjecture: } S_n = \frac{(n+1)! - 1}{(n+1)!}$$

b Proof by Mathematical Induction:

$$P_n : S_n = \frac{(n+1)! - 1}{(n+1)!}, n \in \mathbb{Z}^+.$$

For $n = 1$, show that $\frac{(n+1)! - 1}{(n+1)!} = \frac{n}{(n+1)!}$.

$$\text{LHS: } \frac{(1+1)! - 1}{(1+1)!} = \frac{2! - 1}{2!} = \frac{1}{2!} = \text{RHS}$$

Assume P_k is true, i.e., $S_k = \frac{(k+1)! - 1}{(k+1)!}, k \in \mathbb{Z}^+.$

We want to show that P_{k+1} is true, i.e.,

$$S_{k+1} = \frac{(k+2)! - 1}{(k+2)!}, k \in \mathbb{Z}^+.$$

$$\begin{aligned} S_{k+1} &= S_k + u_{k+1} \\ &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} \text{ by assumption} \\ &= \frac{(k+2)[(k+1)! - 1] + k+1}{(k+2)!} \\ &= \frac{(k+2)! - (k+2) + k+1}{(k+2)!} \\ &= \frac{(k+2)! - 1}{(k+2)!} \\ &= P_{k+1} \end{aligned}$$

Since P_n is true for $n = 1$, and proven true for $n = k+1$ when $n = k, k \in \mathbb{Z}^+$, then by mathematical induction P_{k+1} is true.

b. Using the ratio test, $\lim_{n \rightarrow \infty} \left[\frac{(n+1)}{(n+2)!} \times \frac{(n+1)!}{n} \right] = 0$.

Since $L < 1$, the series converges.

*Finding the sum is best done after Chapter 5 using $f(x) = e^x$ and Taylor series.

Let S denote $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$. Then

$$S + e - 2 = S + \sum_{n=2}^{\infty} \frac{1}{n!} = S + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1.$$

Hence, $S = 1$.

Review Exercise

1 a Using the n th term test for divergence,

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+4} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{4}{n}} \right)^n = \frac{1}{e^4} \neq 0, \text{ hence the}$$

series diverges.

b Using the limit comparison test with $\sum \frac{1}{n}$,

$$\lim_{n \rightarrow \infty} \frac{\frac{\sin \frac{1}{n}}{n}}{\frac{1}{n}} = 1, \text{ hence by comparison with the}$$

harmonic series, the series diverges.

c For $n \geq 2$, and since $2 < e$, $\ln(n) < n - 1$. Hence,

$$\frac{1}{\ln n} > \frac{1}{n-1} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Since the harmonic series diverges, by the comparison test the given series will also diverge.

d By the ratio test,

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{e^{n+1}} \times \frac{e^n}{n} \right) = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \times \frac{e^n}{e^{n+1}} \right) = \frac{1}{e} < 1. \text{ Hence the series converges.}$$

e Since $\cos n\pi = (-1)^n$, $\frac{(n+10)\cos n\pi}{n^{1.4}} = (-1)^n \frac{(n+10)}{n^{1.4}}$.

Since $\lim_{n \rightarrow \infty} \frac{(n+10)}{n^{1.4}} = 0$ and the sequence is decreasing, the given series is convergent by the alternating series test.

2 By the ratio test, $\lim_{n \rightarrow \infty} \left(\frac{k}{(k+1)!} \times \frac{k!}{(k-1)!} \right) = \lim_{n \rightarrow \infty} \frac{k}{k^2 - 1} = 0$.

Hence the series converges. $\sum_{k=1}^{\infty} \frac{(k-1)}{k!} = \sum_{k=1}^{\infty} \frac{1}{k}$.

Furthermore,

$$\begin{aligned} S &= \frac{1}{1!} - \frac{1}{1!} + \frac{2}{2!} - \frac{1}{2!} + \frac{3}{3!} - \frac{1}{3!} + \dots \\ &= 1 - \left(\frac{1}{1!} - \frac{1}{1!} \right) - \left(\frac{1}{2!} - \frac{1}{2!} \right) - \left(\frac{1}{3!} - \frac{1}{3!} \right) - \dots \\ &= 1. \end{aligned}$$

3 $\lim_{n \rightarrow \infty} \frac{1}{(n+1)^7} = 0$, and for all n , $a_n \geq a_{n+1}$, hence the

series converges by the alternating series test 4.

$8^{-7} = 0.000000476837$. The sum of the series therefore, correct to 6d.p., is

$$1 - \frac{1}{2^7} + \frac{1}{3^7} - \dots + \frac{1}{7^7} = 0.992594$$

4 By comparison with the harmonic series,

$$\lim_{n \rightarrow \infty} \left(\frac{1+n}{1+n^2} \times \frac{n}{1} \right) = \lim_{n \rightarrow \infty} \frac{n+n^2}{1+n^2} = 1. \text{ Hence the series}$$

diverges by the limit comparison test.

5 a Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0 \Rightarrow a_n < 1$

for large n . Since $b_n > 0 \Rightarrow a_n b_n < b_n$.

b By comparison, and since $\sum b_n$ converges, then $\sum a_n b_n$ converges. Similarly, $b_n < 1 \Rightarrow b_n^2 < b_n$, and hence $\sum b_n^2$ converges because $\sum b_n$ converges.

6 a $u_n = \frac{3n-2}{2^{\frac{n}{3}}}$. Using the ratio test,

$$\lim_{n \rightarrow \infty} \left(\frac{3n+1}{2^{\frac{n+1}{3}}} \times \frac{2^{\frac{n}{3}}}{3n-2} \right) = 2 \frac{1}{3} \lim_{n \rightarrow \infty} \left(\frac{3n+1}{3n-2} \right) = 2 \frac{1}{3} < 1.$$

Hence the series converges.

b $u_n = \frac{2n-1}{(\sqrt{2})^n}$. Then by the ratio test,

$$\lim_{n \rightarrow \infty} \left(\frac{2n+1}{(\sqrt{2})^{n+1}} \times \frac{(\sqrt{2})^n}{2n-1} \right) = \frac{1}{\sqrt{2}} \lim_{n \rightarrow \infty} \frac{2n+1}{2n-1} = \frac{1}{\sqrt{2}} < 1.$$

Hence the series converges.

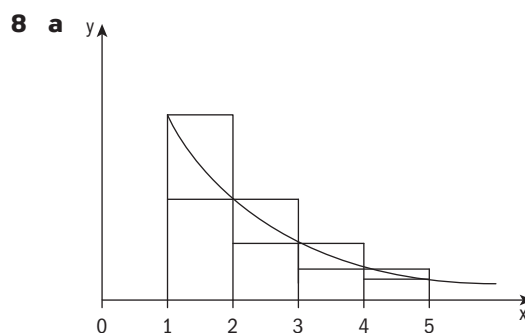
7 Using the limit comparison test with $b_n = \frac{1}{n^2}$,

$$\lim_{n \rightarrow \infty} \left(\frac{1 - \cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \right) = \frac{0}{0}, \text{ an indeterminate form which}$$

allows us to apply L'Hopital's rule. Taking the derivative of numerator and denominator, we

$$\text{obtain } \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n} \times \frac{-1}{n^2}}{\frac{-2}{n^3}} \right) = \frac{1}{2} \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n}}{\frac{1}{n}} \right) = \frac{1}{2} \lim_{n \rightarrow 0} \frac{\sin n}{n} = \frac{1}{2}.$$

Since $\sum \frac{1}{n^2}$ converges by the p-series test, the given series also converges.



The sum of the areas of the smaller rectangles

$$\text{is } \sum_{n=2}^{\infty} \frac{1}{n^p}.$$

The sum of the areas of the larger rectangles is

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Since the area under the curve is between smaller areas and the larger

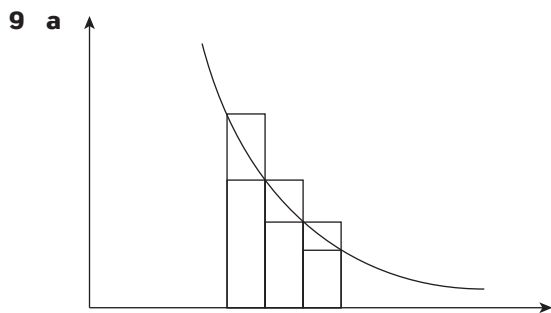
$$\text{areas, } \sum_{n=2}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p} dx < \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

$$\text{b } \int_1^{\infty} \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_1^{\infty} = \frac{1}{p-1}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^p} < 1 + \int_1^{\infty} \frac{1}{x^p} dx = 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

$$\int_1^{\infty} \frac{1}{x^p} dx < \frac{p}{p-1} \text{ and } \int_1^{\infty} \frac{1}{x^p} dx < \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

$$\text{Hence, } \frac{1}{p-1} < \sum_{n=1}^{\infty} \frac{1}{n^p} < \frac{p}{p-1}.$$



The total area of the upper rectangles is

$$\frac{1}{n^4} \times 1 + \frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \dots = \sum_{i=n}^{\infty} \frac{1}{i^4}.$$

The total area of the lower rectangles is

$$\frac{1}{(n+1)^4} \times 1 + \frac{1}{(n+2)^4} \times 1 + \frac{1}{(n+3)^4} \times 1 + \dots = \sum_{i=n+1}^{\infty} \frac{1}{i^4}.$$

The total area under the curve from $x = n$ to infinity lies between the above two sums, hence

$$\sum_{i=n+1}^{\infty} \frac{1}{i^4} < \int_n^{\infty} \frac{1}{x^4} dx < \sum_{i=n}^{\infty} \frac{1}{i^4}.$$

b $\int_n^{\infty} \frac{1}{x^4} dx = -\left[\frac{1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3}$. Hence, $\sum_{i=n+1}^{\infty} \frac{1}{i^4} < \frac{1}{3n^3}$.

Then, adding $\sum_{i=1}^n \frac{1}{i^4}$ to both sides of the

inequality, $S < \sum_{i=1}^n \frac{1}{i^4} + \frac{1}{3n^3}$. In the same way

$$\sum_{i=n}^{\infty} \frac{1}{i^4} > \frac{1}{3n^3}, \text{ and adding } \sum_{i=1}^{n-1} \frac{1}{i^4} \text{ to both sides,}$$

$$S > \sum_{i=1}^{n-1} \frac{1}{i^4} + \frac{1}{3n^3}.$$

Hence, the value of S lies between $\sum_{i=1}^n \frac{1}{i^4} + \frac{1}{3n^3}$ and $\sum_{i=1}^{n-1} \frac{1}{i^4} + \frac{1}{3n^3}$.

c i When $n = 8$, $S < 1.08243\dots$ and $S > 1.08219\dots$, hence $S = 1.082$ to 3 d.p.

ii Substituting this value for S ,

$$N \approx \frac{\pi^4}{1.082} \approx 90.0268\dots \Rightarrow N = 90.$$

10 a $S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > S_n + \frac{1}{2n} + \frac{1}{2n} + \dots + \frac{1}{2n} = S_n + \frac{1}{2}.$

b Replacing n by $2n$, $S_{4n} > S_{2n} + \frac{1}{2} > S_n + 1$.

Continuing this way, $S_{8n} > S_n + \frac{3}{2}$. And in

general, $S_{2^m n} > S_n + \frac{m}{2}$. Then, letting $n = 2$,

$$S_{2^{m+1}} > S_2 + \frac{m}{2}.$$

c $S_{2^{m+1}} > N$ if $S_2 + \frac{m}{2} > N$, or if $m > 2(N - S_2)$. Hence the sequence is divergent.

11 i The sequence of absolute values is decreasing and the n th term tends to 0 as n tends to infinity, hence by the alternating series test, the series converges.

ii The partial sums are 0.333, 0.111, 0.269, 0.148, 0.246. S_{∞} lies between any pair of successive partial sums, hence it lies between 0.148 and 0.246, therefore it is less than 0.25.

12 a The sequence of absolute values is decreasing and the n th term tends to 0 as n tends to infinity, hence by the alternating series test, the series converges.

b $S_4 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = 0.841468(6 \text{ d.p.}).$

S_4 error $< a_5$, hence

$$\text{error} < \frac{1}{9!}, \text{ or error} < 0.00000276.$$

5

Everything polynomic

Exercise 5A

a For $f(x) = \frac{1}{1-4x}$, $r = 4x$,

$$|4x| < 1 \Rightarrow |x| < \frac{1}{4}, -\frac{1}{4} < x < \frac{1}{4}, x \in \left] -\frac{1}{4}, \frac{1}{4} \right[.$$

In this interval,

$$\frac{1}{1-4x} = 1 + 4x + (4x)^2 + \dots + (4x)^n + \dots = \sum_{n=0}^{\infty} (4x)^n,$$

$$\text{and } R = \frac{1}{4}.$$

b For $f(x) = \frac{1}{1+5x}$, $r = -5x$; $|r| < 1 \Rightarrow |x| < \frac{1}{5}$,

$$-\frac{1}{5} < x < \frac{1}{5}, x \in \left] -\frac{1}{5}, \frac{1}{5} \right[. \text{ In this interval,}$$

$$\begin{aligned} \frac{1}{1-(-5x)} &= 1 + (-5x) + (-5x)^2 + (-5x)^3 + \dots + (-5x)^n + \dots \\ &= 1 - 5x + 5^2 x^2 + \dots + (-1)^n (5x)^n + \dots \end{aligned}$$

$$\text{Hence, } \frac{1}{1+5x} = \sum_{n=0}^{\infty} (-1)^n (5x)^n; R = \frac{1}{5}.$$

c $f(x) = \frac{1}{x-4} = \frac{1}{4(\frac{1}{4}x-1)} = \frac{1}{-4(1-\frac{1}{4}x)}.$

$$\left| \frac{1}{4}x \right| < 1 \Rightarrow |x| < 4, -4 < x < 4, x \in \left] -4, 4 \right[.$$

$$\text{In this interval, } \frac{1}{x-4} = -\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}x \right)^n; R = 4$$

d $f(x) = \frac{1}{3x} = \frac{1}{3} \left(\frac{1}{x} \right) = \frac{1}{3} \left(\frac{1}{1+(x-1)} \right)$
 $= \frac{1}{3} \left(\frac{1}{1-[-(x-1)]} \right); |x-1| < 1 \Rightarrow 0 < x < 2.$

$$\text{Hence, the interval is } \left] 0, \frac{2}{3} \right[$$

$$\text{In this interval, } f(x) = \sum_{n=0}^{\infty} (-1)^n (x-1)^n; R = \frac{1}{3}.$$

e $f(x) = \frac{x}{1+x}; |x| < 1 \Rightarrow -1 < x < 1, x \in \left] -1, 1 \right[.$ In this interval,

$$\begin{aligned} \frac{x}{1+x} &= x(1 + (-x) + (-x)^2 + \dots + (-x)^n + \dots) \\ &= x \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+1}; R = 1 \end{aligned}$$

f $f(x) = \frac{2}{1+4x}; |-4x| < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}, x \in \left] -\frac{1}{4}, \frac{1}{4} \right[.$

In this interval,

$$\begin{aligned} \frac{2}{1+4x} &= 2(1 + (-4x) + (-4x)^2 + \dots + (-4x)^n + \dots) \\ &= 2 \sum_{n=0}^{\infty} (-1)^n (4x)^n = \sum_{n=0}^{\infty} 2 \cdot (-1)^n (4x)^n; R = \frac{1}{4}. \end{aligned}$$

Exercise 5B

a Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \frac{x^{n+1}}{n+1}}{(-1)^n \frac{x^n}{n}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n}{n+1} = |x|.$

The series converges in the interval $|x| < 1$, hence

$R = 1$. At $x = 1$, $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$ is the alternating harmonic

series, which converges conditionally. At $x = -1$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Hence, the interval of convergence is $] -1, 1[$, and $R = 1$.

b Using the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Hence, the interval of convergence is the set of Reals, and $R = \text{infinity}$.

c Using the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{2^{n+1} x^{n+1}}{(n+2)^2}}{\frac{2^n x^n}{(n+1)^2}} \right| = 2|x| \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+2)^2} = 2|x|. \text{ Hence, the series}$$

converges in the interval $2|x| < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$. At

$$x = -\frac{1}{2}, \sum_{n=0}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}.$$

Since the absolute value of the series is decreasing, and $\lim_{n \rightarrow \infty} u_n = 0$,

the series converges conditionally. At $x = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}.$$

This series converges by comparison with $\sum \frac{1}{n^2}$, which converges by the

p-series test. Hence, the interval of convergence is

$$\left] -\frac{1}{2}, \frac{1}{2} \right], \text{ and } R = \frac{1}{2}.$$

d Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{(-1)^n \frac{(2x+3)^{n+1}}{(n+1) \ln(n+1)}}{(-1)^n \frac{(2x+3)^n}{n \ln n}} \right|$
 $= |2x+3| \lim_{n \rightarrow \infty} \left| \frac{n \ln n}{(n+1) \ln(n+1)} \right| = |2x+3|. \text{ The interval}$

of convergence is $|2x+3| < 1$, or $-2 < x < -1$.

At $x = -2$, $\sum \frac{1}{n \ln n}$ is divergence by comparison with

the harmonic series, and when $x = -1$, the series converges conditionally. Hence, the interval of

convergence is $] -2, -1[$, and $R = \frac{1}{2}$.

e Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(x-1)^{n+2}}{n+2} \cdot \frac{n+1}{(-1)^{n+1}(x-1)^{n+1}} \right|$
 $= |-1(x-1)| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| = |x-1|$. The interval of convergence is $|x-1| < 1$, or $0 < x < 2$. At $x = 0$, $\sum (-1)^{n+1} \frac{1}{n+1}$ diverges by comparison with $\frac{1}{\sqrt{n}}$, which diverges by the p-series test. At $x = 2$, $\sum (-1)^{n+1} \frac{1}{n+1}$ converges conditionally by the alternating series test. Hence, the interval of convergence is $]0, 2]$, and $R = 1$.

f Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = x^2 \lim_{n \rightarrow \infty} \frac{(2n+3)!}{(2n+1)!}$
 $= x^2 \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} = x^2 \cdot 0 = 0$. Hence, the interval of convergence is the set of Reals, and $R = \text{infinity}$.

g Using the ratio test,
 $\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)\sqrt{n+1} \cdot 3^n} \cdot \frac{n\sqrt{n}}{x^n \cdot 3^{n-1}} \right| = |x| \lim_{n \rightarrow \infty} \frac{n\sqrt{n}}{(n+1)\sqrt{n+1} \cdot 3}$
 $= \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{\sqrt{n^3}}{\sqrt{(n+1)^3}} = \frac{|x|}{3}$.

The interval of convergence is $\frac{|x|}{3} < 1 \Rightarrow |x| < 3$.

At $x = 3$, $\sum_{n=1}^{\infty} \frac{3n}{n\sqrt{n} \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by

p-series test. At $x = -3$, $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n} \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$, the

series converges conditionally by the alternating series test. Hence the interval of convergence is $[-3, 3]$, and $R = 3$.

h Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} \cdot x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n} \cdot x^n} \right| = \frac{|x|}{3} \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n}}$
 $= \frac{|x|}{3} \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n}} = \frac{|x|}{3}$. The interval of convergence is $\frac{|x|}{3} < 1 \Rightarrow |x| < 3$. At $x = 3$, $\sum_{n=1}^{\infty} \sqrt{n}$ diverges, and at $x = -3$, $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ also diverges. Hence, the interval of convergence is $] -3, 3[$, and $R = 3$.

i Using the ratio test,
 $\lim_{n \rightarrow \infty} \left| \frac{\sin^{n+1} x}{(n+1)^2} \cdot \frac{n^2}{\sin^n x} \right| = |\sin x| \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = |\sin x|$

The interval of convergence is $|\sin(x)| < 1$, and this is true for all x . Hence, the interval of convergence is the set of Reals, and $R = \text{infinity}$.

j Using the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{e^{(n+1)x}}{n+1} \cdot \frac{n}{e^{nx}} \right| = e^x \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = e^x$

Hence, the interval of convergence is $e^x < 1$, or $x < 0$, and $R = \text{infinity}$.

k Using the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{e^{(n+1)x}}{n+1} \cdot \frac{n}{e^{nx}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x+1}{3} \right| = \frac{x+1}{3}.$$

This converges for $\left| \frac{x+1}{3} \right| < 1 \Rightarrow -4 < x < 2$.

At $x = -4$, $\sum_{n=1}^{\infty} \frac{(-3)^n}{3} = \sum_{n=1}^{\infty} (-1)^n$, which diverges. At

$x = 2$, $\sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1$ which also diverges. Hence, the interval of convergence is $] -4, 2[$, and $R = 3$.

2 a Applying the ratio test,

$$\lim_{n \rightarrow \infty} \left| \frac{(kx)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(kx)^n} \right| = |x| \lim_{n \rightarrow \infty} \left| k \cdot \frac{n^2}{(n+1)^2} \right| = k|x|.$$

$$k|x| < 1 \Rightarrow |x| < \frac{1}{k}. \text{ Since } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow k = 2.$$

b Applying the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{(x-k)^{2(n+1)}}{(n+1)^4} \cdot \frac{n^4}{(x-k)^{2n}} \right|$
 $= (x-k)^2 \lim_{n \rightarrow \infty} \frac{n^4}{(n+1)^4} = (x-k)^2$. The interval

of convergence is $(x-k)^2 < 1 \Rightarrow -1 < x-k < 1$
 $\Rightarrow -1+k < x < 1+k$. Since the interval of convergence is given as $[2, 4] \Rightarrow -1+k = 2$
and $k+1 = 4$. Hence, $k = 3$.

3 Applying the ratio test, $\lim_{n \rightarrow \infty} \left| \frac{(x-h)^{n+1}}{(n+1)k^{n+1}} \cdot \frac{n k^n}{(x-h)^n} \right|$
 $= \frac{|x-h|}{k} \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = \frac{|x-h|}{k}$. Since $\frac{|x-h|}{k} < 1$
 $\Rightarrow |x-h| < k \Rightarrow -k < x-h < k \Rightarrow -k+h < x < k+h$
Since the interval of convergence is $] -1, 7[\Rightarrow -k+h = 1$ and $k+h = 7$. Solving simultaneously, $h = 4$ and $k = 3$.

Exercise 5C

1 $\frac{2}{1-x^2} = 2(1+x^2+x^4+\dots+x^{2n}+\dots) = 2\sum_{n=0}^{\infty} x^{2n}$.
 $I =]-1, 1[$, $R = 1$.

2 $\frac{1}{(x+1)^2} = \frac{1}{1+x} \cdot \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \cdot \sum_{n=0}^{\infty} (-1)^n x^n$
 $= \sum_{n=0}^{\infty} (-1)^n (n+1)x^n$. $I =]-1, 1[$; $R = 1$.

3 $\frac{1}{4x^2+1} = 1-4x^2+(4x^2)^2-(4x^2)^3+\dots$
 $= \sum_{n=0}^{\infty} (-1)^n (4x^2)^n = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$.
 $I = \left]-\frac{1}{2}, \frac{1}{2}\right[$; $R = \frac{1}{2}$.

- 4 This example should be done in 5F using the Binomial Series. The solution is shown here.

$$\begin{aligned}\sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\&= 1 + \frac{1}{2}x + \frac{1}{2} \times \left(-\frac{1}{2}\right) \frac{x^2}{2!} + \frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \frac{x^3}{3!} + \dots \\&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots; -1 < x < 1 \\ \frac{1}{\sqrt{1-x}} &= (1-x)^{-\frac{1}{2}} \\&= 1 - \left(-\frac{1}{2}\right)x + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \frac{x^2}{2!} - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \frac{x^3}{3!} + \dots \\&= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots, -1 < x < 1. \\ (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} \\&= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots\right) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots\right) \\&= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} \dots;]-1, 1[; R = 1\end{aligned}$$

5 $\ln x = \int \frac{1}{x}; \frac{1}{x} = \frac{1}{1+(x-1)}$
 $= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$

$$\begin{aligned}\int_1^x \frac{1}{1+(t-1)} dt &= \int_1^x (1 - (t-1) + (t-1)^2 - (t-1)^3 + \dots) dt \\&= \left[t - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \dots \right]_1^x \\&= \left(x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots \right) - 1 \\&= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\ I = |x-1| < 1 &\Rightarrow I =]0, 2[; R = 1\end{aligned}$$

6 $\sum_{n=0}^{\infty} (a_n x^n + b_n x^n)$ is convergent when $|x| > 1$,
hence $R = 1$.

7 For $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$, $\lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} \times \frac{1}{2} x \right| = \left| \frac{x}{2} \right|$ which is
convergent for $|x| < 2$, so $R = 2$. Testing the end-
points, for $x = 2$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series
test. For $x = -2$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges absolutely.
Hence, the interval of convergence is $[-2, 2]$
For $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2 2^n}$, $\lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \times \frac{1}{2} x \right| = \left| \frac{x}{2} \right|$, which is also
convergent for $|x| < 2$. Testing the end points, at
 $x = 2$, $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and at $x = -2$, $\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$
converges conditionally.

Since $\frac{nx^{n-1}}{n^2 2^n}$ is the derivative of $\frac{x^n}{n^2 2^n}$, the interval of
convergence would be the same, except for possibly
the end points. Hence, the interval of convergence is
 $[-2, 2[$.

8 $\frac{2}{x+2} + \frac{1}{x-1} = \frac{2(x-1) + (x+2)}{(x+2)(x-1)} = \frac{3x}{x^2 + x - 2}$

$$\begin{aligned}\frac{2}{x+2} &= \sum_{n=0}^{\infty} \frac{1}{1 - \left(-\frac{1}{2}x\right)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}x\right)^n, I =]-2, 2[; \\ R &= 2. \\ \frac{1}{x-1} &= \sum_{n=0}^{\infty} \frac{-1}{1-x} = -1 \sum_{n=0}^{\infty} \frac{1}{1-x} = \sum_{n=0}^{\infty} -x^n; \\ I &=]-1, 1[; R = 1. \\ \frac{3x}{x^2 + x - 2} &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty} -x^n = \sum_{n=0}^{\infty} \left[\frac{1}{(-2)^n} - 1 \right] x^n.\end{aligned}$$

Hence, $I =]-1, 1[; R = 1$.

Exercise 5D

1 a $f(x) = \frac{1}{1-x} \Rightarrow f(0) = 1$,
 $f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$,
 $f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2, f^{(3)}(x) = \frac{6}{(1-x)^4}$
 $\Rightarrow f^{(3)}(0) = 6$,
 $f^{(4)}(x) = \frac{24}{(1-x)^5} \Rightarrow f^{(4)}(0) = 24$
 $T_4(x) = \frac{1}{0!} + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!}$
 $= 1 + x + x^2 + x^3 + x^4$

b $f(x) = x \sin x \Rightarrow f(0) = 0$,
 $f'(x) = \sin x + x \cos x \Rightarrow f'(0) = 0$,
 $f''(x) = \cos x + \cos x - x \sin x$
 $= 2 \cos x - x \sin x \Rightarrow f''(0) = 2$,
 $f^{(3)}(x) = -2 \sin x - \sin x - x \cos x$
 $= -3 \sin x - x \cos x \Rightarrow f^{(3)}(0) = 0$,
 $f^{(4)}(x) = -3 \cos x - \cos x + x \sin x$
 $= -4 \cos x + x \sin x \Rightarrow f^{(4)}(0) = -4$
 $T_4(x) = \frac{2x^2}{2!} + \frac{-4x^4}{4!} = x^2 - \frac{1}{6}x^4$

c $f(x) = (1+x)^{\frac{1}{3}} \Rightarrow f(0) = 1$,
 $f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}} \Rightarrow f'(0) = \frac{1}{3}$,
 $f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}} \Rightarrow f''(0) = -\frac{2}{9}$,

$$f^{(3)}(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}} \Rightarrow f^{(3)}(0) = \frac{10}{27},$$

$$f^{(4)}(x) = -\frac{80}{81}(1+x)^{-\frac{8}{3}} \Rightarrow f^{(4)}(0) = -\frac{80}{81}$$

$$\begin{aligned} T_4(x) &= \frac{1}{0!} + \frac{\frac{1}{3}x}{1!} + \frac{-\frac{2}{9}x^2}{2!} + \frac{\frac{10}{27}x^3}{3!} + \frac{-\frac{80}{81}x^4}{4!} \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 \end{aligned}$$

d $f(x) = xe^x \Rightarrow f(0) = 0, f'(x) = e^x + xe^x$
 $\Rightarrow f'(0) = 1,$

$$f''(x) = 2e^x + xe^x \Rightarrow f''(0) = 2,$$

$$f^{(3)}(x) = 3e^x + xe^x \Rightarrow f^{(3)}(0) = 3,$$

$$f^{(4)}(x) = 4e^x + xe^x \Rightarrow f^{(4)}(0) = 4$$

$$T_4(x) = \frac{x}{1!} + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

2 a $f(x) = \frac{1}{x} \Rightarrow f(\pi) = \frac{1}{\pi}, f'(x) = -\frac{1}{x^2}$

$$\Rightarrow f'(\pi) = -\frac{1}{\pi^2}, f''(x) = \frac{2}{x^3}$$

$$\Rightarrow f''(\pi) = \frac{2}{\pi^3}, f^{(3)}(x) = -\frac{6}{x^4}$$

$$\Rightarrow f^{(3)}(\pi) = -\frac{6}{\pi^3}$$

$$\begin{aligned} T_3(x) &= \frac{1}{0!} + \frac{-\frac{1}{\pi^2}(x-\pi)}{1!} + \frac{\frac{2}{\pi^3}(x-\pi)^2}{2!} + \frac{-\frac{1}{\pi^2}(x-\pi)^3}{3!} \\ &= \frac{1}{\pi} - \frac{1}{\pi^2}(x-\pi) + \frac{2}{\pi^3}(x-\pi)^2 - \frac{1}{\pi^4}(x-\pi)^3 \end{aligned}$$

b $f(x) = x^{-1} \sin x \Rightarrow f(\pi) = 0,$

$$f'(x) = -x^{-2} \sin x + x^{-1} \cos x \Rightarrow f'(\pi) = \frac{1}{\pi},$$

$$\begin{aligned} f''(x) &= 2x^{-3} \sin x - x^{-2} \cos x - x^{-2} \cos x - x^{-1} \sin x \\ &= 2x^{-3} \sin x - 2x^{-2} \cos x - x^{-1} \sin x \end{aligned}$$

$$\Rightarrow f''(\pi) = -\frac{2}{\pi^2}$$

$$\begin{aligned} f^{(3)}(x) &= -6x^{-4} \sin x + 2x^{-3} \cos x + 4x^{-3} \cos x \\ &\quad + 2x^{-2} \sin x + x^{-2} \sin x - x^{-1} \cos x \end{aligned}$$

$$= -6x^{-4} \sin x + 6x^{-3} \cos x + 3x^{-2} \sin x - x^{-1} \cos x$$

$$\Rightarrow f^{(3)}(\pi) = -\frac{6}{\pi^3} + \frac{1}{\pi}$$

$$\begin{aligned} T_3(x) &= \frac{-\frac{1}{\pi}(x-\pi)}{1!} + \frac{\frac{2}{\pi^2}(x-\pi)^2}{2!} + \frac{\left(-\frac{6}{\pi^3} + \frac{1}{\pi}\right)(x-\pi)^3}{3!} \\ &= -\frac{1}{\pi}(x-\pi) + \frac{1}{\pi^2}(x-\pi)^2 + \left(\frac{1}{6\pi} - \frac{1}{\pi^3}\right)(x-\pi)^3 \end{aligned}$$

c $f(x) = x^2 \cos x \Rightarrow f(\pi) = -\pi^2,$

$$f'(x) = 2x \cos x - x^2 \sin x \Rightarrow f'(\pi) = -2\pi,$$

$$f''(x) = 2 \cos x - 2x \sin x - 2x \sin x - x^2 \cos x$$

$$= (2 - x^2) \cos x - 4x \sin x$$

$$\Rightarrow f''(\pi) = \pi^2 - 2$$

$$f^{(3)}(x) = -2x \cos x - (2 - x^2) \sin x - 4 \sin x - 4x \cos x$$

$$= -6x \cos x - (x^2 + 6) \sin x$$

$$\Rightarrow f^{(3)}(\pi) = -6\pi$$

$$T_3(x) = \frac{-\pi^2}{0!} - \frac{2\pi(x-\pi)}{1!} + \frac{(\pi^2-2)(x-\pi)^2}{2!} + \frac{-6\pi(x-\pi)^3}{3!}$$

$$= -\pi^2 - 2\pi(x-\pi) + \left(\frac{\pi^2}{2} - 1\right)(x-\pi)^2 + \pi(x-\pi)^3$$

3 $f(x) = 2x^3 - 3x^2 + 4x - 5 \Rightarrow f'(x) = 6x^2 - 6x + 4$

$$\Rightarrow f''(x) = 12x - 6 \Rightarrow f^{(3)}(x) = 12$$

a $f(0) = -5, f'(0) = 4, f''(0) = -6, f^{(3)}(0) = 12$

$$\begin{aligned} \Rightarrow T_3(x) &= \frac{-5}{0!} + \frac{4x}{1!} + \frac{-6x^2}{2!} + \frac{12x^3}{3!} \\ &= -5 + 4x - 3x^2 + 2x^3 \end{aligned}$$

b $f(1) = 2 - 3 + 4 - 5 = -2, f'(1) = 6 - 6 + 4 = 4,$

$$f''(1) = 12 - 6 = 6, f^{(3)}(1) = 12 \Rightarrow$$

$$\begin{aligned} T_3(x) &= \frac{-2}{0!} + \frac{4(x-1)}{1!} + \frac{6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!} \\ &= -2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3 \end{aligned}$$

4 $T_1(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} = f(a) + f'(a)(x-a)$

Equation of a tangent to the curve $y = f(x)$ at the point where $x = a$ is obtained in the following way:

Point $(a, f(a))$, slope $m = f'(a)$

$$\text{Equation: } y - y_1 = m(x - x_1) \Rightarrow y - f(a)$$

$$= f'(a)(x - a) \Rightarrow y = f(a) + f'(a)(x - a)$$

5 $f(1) = 4, f'(1) = -1, f''(1) = 3, f^{(3)}(1) = 2 \Rightarrow$

$$T_3(x) = \frac{4}{0!} + \frac{-1(x-1)}{1!} + \frac{3(x-1)^2}{2!} + \frac{2(x-1)^3}{3!}$$

$$= 4 - (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{4}(x-1)^3$$

$$\begin{aligned} f(1.2) &\approx T_3(1.2) = 4 - 0.2 + \frac{3}{2} \times 0.2^2 + \frac{1}{4} \times 0.2^3 \\ &= 3.86267 \end{aligned}$$

Exercise 5E

- 1** $f(x) = \cos x \Rightarrow f'(x) = -\sin x$
 $\Rightarrow f''(x) = -\cos x \Rightarrow f^{(3)}(x) = \sin x \Rightarrow f^{(4)}(x) = \cos x$
 We can summarize this into the following form
 (similar to the powers of the imaginary unit i):

$$f^{(n)}(x) = \begin{cases} -\sin x, & n = 4k - 3 \\ -\cos x, & n = 4k - 2 \\ \sin x, & n = 4k - 1 \\ \cos x, & n = 4k \end{cases}, k \in \mathbb{Z}^+$$

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} \Rightarrow |f^{(n+1)}(t)| \leq 1$$

$$0 < \frac{|f^{(n+1)}(t)|}{(n+1)!} |x^{n+1}| \leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Rightarrow \lim_{n \rightarrow \infty} R_n = 0$$

$$\text{Hence } \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x \in \mathbb{R}.$$

2 a $f(x) = \sin^2 x = \sin x \times \sin x$

$$\begin{aligned} &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\ &= x^2 - \frac{2x^4}{3!} + \left(\frac{2}{5!} + \frac{1}{(3!)^2} \right) x^6 - \left(\frac{2}{7!} + \frac{2}{3! \times 5!} \right) x^8 + \dots \\ &= \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{12+20}{6!} x^6 - \frac{16+112}{8!} x^8 + \dots \\ &= \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32}{6!} x^6 - \frac{128}{8!} x^8 + \dots + (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!} + \dots \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!} \end{aligned}$$

- b** We use the expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ where instead of x we input $3x$

$$\begin{aligned} f(x) &= e^{3x} = \frac{(3x)^0}{0!} + \frac{(3x)^1}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots \\ &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \dots + \frac{3^n x^n}{n!} + \dots \end{aligned}$$

c $f(x) = \ln(1-x) \Rightarrow f'(x) = -\frac{1}{1-x}$
 $\Rightarrow f''(x) = -\frac{1}{(1-x)^2} \Rightarrow f'''(x) = -\frac{2}{(1-x)^3}$

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}, n \in \mathbb{Z}^+$$

$$f(0) = \ln(1) = 0 \Rightarrow f'(0) = -1 \Rightarrow f''(0) = -1$$

$$\Rightarrow f'''(0) = -2$$

$$f^{(n)}(0) = -(n-1)!, n \in \mathbb{Z}^+$$

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} -\frac{(n-1)!}{n!} x^n = \sum_{n=1}^{\infty} -\frac{x^n}{n} \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} - \dots \end{aligned}$$

d $f(x) = \tan x \Rightarrow f'(x) = (\cos x)^{-2}$
 $\Rightarrow f''(x) = -2(\cos x)^{-3} \times (-\sin x) = 2(\cos x)^{-3} \sin x$
 $\Rightarrow f^{(3)}(x) = -6(\cos x)^{-4} \times (-\sin x) \times \sin x$
 $\quad + 2(\cos x)^{-3} \times (-\cos x)$
 $\quad = 2(\cos x)^{-4} (1 + 2(\sin x)^2)$
 $\Rightarrow f^{(4)}(x) = -8(\cos x)^{-5} \times (-\sin x)(1 + 2(\sin x)^2)$
 $\quad + 2(\cos x)^{-4} 4 \sin x \cos x$
 $\quad = 8 \sin x (\cos x)^{-5} (2 + (\sin x)^2)$
 $\quad = 8(\cos x)^{-5} (2 \sin x + (\sin x)^3)$
 $\Rightarrow f^{(5)}(x) = -40(\cos x)^{-6} \times (-\sin x)(2 \sin x + (\sin x)^3)$
 $\quad + 8(\cos x)^{-5} (2 \cos x + 3(\sin x)^2 \cos x)$

$$\begin{aligned} &= 8(\cos x)^{-6} (10(\sin x)^2 + 5(\sin x)^4 + 2(\cos x)^2 \\ &\quad + 3(\sin x)^2 (\cos x)^2) \\ &= 8(\cos x)^{-6} (10(\sin x)^2 + 5(\sin x)^4 + 2(1 - (\sin x)^2) \\ &\quad + 3(\sin x)^2 (1 - (\sin x)^2)) \\ &= \frac{16(\sin x)^4 + 88(\sin x)^2 + 16}{(\cos x)^6} \end{aligned}$$

$$f(0) = 0 \Rightarrow f'(0) = 1 \Rightarrow f''(0) = 0 \Rightarrow$$

$$f^{(3)}(0) = 2 \Rightarrow f^{(4)}(0) = 0 \Rightarrow f^{(5)}(0) = 16 \Rightarrow$$

$$\begin{aligned} f(x) &= x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \dots \\ &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \end{aligned}$$

- e The derivatives are best found using a computer algebra system.

$$f(x) = \arctan(x^2); f(0) = 0$$

$$f'(x) = \frac{2x}{x^4 + 1}; f'(0) = 0$$

$$f''(x) = \frac{-2(3x^4 - 1)}{(x^4 + 1)^2}; f''(0) = 2$$

$$f'''(x) = \frac{8x^3(3x^4 - 5)}{(x^4 + 1)^3}; f'''(0) = 0$$

$$f^{(4)}(x) = \frac{-24x^2(5x^8 - 22x^4 + 5)}{(x^4 + 1)^4}; f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \frac{48x(15x^{12} - 135x^8 + 101x^4 - 5)}{(x^4 + 1)^5};$$

$$f^{(5)}(0) = 0$$

$$f^{(6)}(x) = \frac{-240(21x^{16} - 336x^{12} + 546x^8 - 120x^4 + 1)}{(x^4 + 1)^6};$$

$$f^{(6)}(0) = -240$$

Hence,

$$f(x) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots + (-1)^n \frac{x^{4n+2}}{2n+1} + \dots$$

f $f(x) = e^x \times \sin x$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{5!} - \frac{1}{2! \times 3!} + \frac{1}{4!}\right)x^5 + \dots$$

$$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

3 $f(x) = \sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$

$$= \frac{1}{2} - \frac{1}{2} \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \dots = \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!}$$

4 $f(x) = \frac{\sin x}{x} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \times \frac{1}{x}$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots\right)$$

$$= 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \dots = 1$$

- 5 We use the expansion of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ where instead of x we input ix and $-ix$ respectively.

1 $e^{ix} = \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!}$

$$+ \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{2!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots$$

2 $e^{-ix} = \frac{(-ix)^0}{0!} + \frac{(-ix)^1}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!}$

$$+ \frac{(-ix)^6}{6!} + \frac{(-ix)^7}{7!} + \dots$$

$$= 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{2!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \dots$$

- a If we subtract equation (2) from equation (1) we obtain the following:

$$e^{ix} - e^{-ix} = \left(1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots\right)$$

$$- \left(1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \dots\right)$$

$$= 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \dots$$

$$= 2i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$= 2i \sin x \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

- b If we add equations (1) and (2) we obtain the following:

$$e^{ix} + e^{-ix} = \left(1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots\right)$$

$$+ \left(1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \dots\right)$$

$$= 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \dots$$

$$= 2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$= 2 \cos x \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

- 6 We can solve this question by using the results from the previous question directly or using algebra.

Method I

$$\begin{aligned}
 e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots \\
 &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right) \\
 &= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos \theta} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right)}_{\sin \theta} \\
 &= \cos \theta + i \sin \theta
 \end{aligned}$$

Method II

$$\begin{aligned}
 \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \Rightarrow \cos \theta + i \sin \theta \\
 &= \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{2e^{i\theta}}{2} = e^{i\theta}
 \end{aligned}$$

Exercise 5F

1 a $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-x)^n$, we are

going to simplify the binomial coefficient

$$\begin{aligned}
 \binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \times \dots \times \left(\frac{1}{2} - n + 1\right)}{n!} \\
 &= (-1)^{n-1} \frac{1 \times 3 \times 5 \times \dots \times (2n-3)}{2^n \times n!} \\
 &= (-1)^{n-1} \frac{(2n)!}{2^n \times n! \times 2^n \times n! \times (2n-1)} \\
 &= (-1)^{n-1} \frac{(2n)!}{(2^n \times n!)^2 (2n-1)}, \text{ so} \\
 &= 1 + \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2n)!}{(2^n \times n!)^2 (2n-1)} (-1)^n x^n \\
 &= 1 - \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n \times n!)^2 (2n-1)} x^n \\
 |x| < 1 &\Rightarrow x \in]-1, 1[
 \end{aligned}$$

b $f(x) = \frac{1}{(1+x)^3} = (1+x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} x^n$, we are

going to simplify the binomial coefficient

$$\begin{aligned}
 \binom{-3}{n} &= \frac{(-3) \times (-4) \times \dots \times (-3-n+1)}{n!} \\
 &= (-1)^n \frac{3 \times 4 \times \dots \times (n+2)}{n!} = (-1)^n \frac{(n+2)!}{2 \times n!} \\
 &= (-1)^n \frac{(n+1)(n+2)}{2} = (-1)^n \left(\frac{n+2}{2} \right), \text{ so} \\
 &= \sum_{n=0}^{\infty} (-1)^n \binom{n+2}{2} x^n \\
 &= 1 - 3x + 6x^2 - 10x^3 + 21x^4 - 28x^5 + \dots \\
 |x| < 1 &\Rightarrow x \in]-1, 1[
 \end{aligned}$$

c $f(x) = \frac{1}{(1-4x^2)^2} = (1-4x^2)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-4x^2)^n$,

we can simplify this binomial coefficient

$$\begin{aligned}
 \binom{-2}{n} &= \frac{(-2) \times (-3) \times \dots \times (-2-n+1)}{n!} \\
 &= (-1)^n \frac{2 \times 3 \times \dots \times (n+1)}{n!} \\
 &= (-1)^n \frac{(n+1)!}{n!} = (-1)^n (n+1),
 \end{aligned}$$

$$\begin{aligned}
 \text{so } &\sum_{n=0}^{\infty} (-1)^n (n+1) (-4)^n x^{2n} = \sum_{n=0}^{\infty} 4^n (n+1) x^{2n} \\
 &= 1 + 8x^2 + 48x^4 + 256x^6 + \dots
 \end{aligned}$$

$$|-4x^2| < 1 \Rightarrow x^2 < \frac{1}{4} \Rightarrow x \in \left] -\frac{1}{2}, \frac{1}{2} \right[$$

d $f(x) = \frac{1}{\sqrt[4]{1+2x^3}} = (1+2x^3)^{-\frac{1}{4}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{4}}{n} (2x^3)^n$,

we are going to the binomial coefficient

$$\begin{aligned}
 \binom{-\frac{1}{4}}{n} &= \frac{\left(-\frac{1}{4}\right) \times \left(-\frac{5}{4}\right) \times \left(-\frac{9}{4}\right) \times \dots \times \left(-\frac{1}{4} - n + 1\right)}{n!} \\
 &= (-1)^n \frac{1 \times 5 \times 9 \times \dots \times (4n-3)}{4^n \times n!}, \text{ so} \\
 &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 5 \times 9 \times \dots \times (4n-3)}{4^n \times n!} (2x^3)^n \\
 &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \times 5 \times 9 \times \dots \times (4n-3)}{2^n \times n!} x^{3n} \\
 |2x^3| < 1 &\Rightarrow |x^3| < \frac{1}{2} \Rightarrow |x| < \frac{1}{\sqrt[3]{2}}, x \in \left] -\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} \right[
 \end{aligned}$$

2 $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

$$= (1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$$

$$\begin{aligned}
 \binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \times \dots \times \left(-\frac{1}{2} - n + 1\right)}{n!} \\
 &= (-1)^n \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2^n \times n!} \\
 &= (-1)^n \frac{(2n)!}{2^n \times n! \times 2^n \times n!} = (-1)^n \frac{(2n)!}{(2^n \times n!)^2}, \text{ so} \\
 &= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{(2^n \times n!)^2} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n \times n!)^2} x^{2n} \\
 f(x) &= \int \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n \times n!)^2} x^{2n} dx
 \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n \times n!)^2} \frac{x^{2n+1}}{2n+1} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$$

$$\begin{aligned} 3 \quad a \quad \sqrt{9+x} &= 3\sqrt{1+\frac{x}{9}} = 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}} = 3 \times \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \left(\frac{x}{9}\right)^n \\ &= 3\left(1 + \frac{1}{2} \times \frac{x}{9} - \frac{1}{8} \times \frac{x^2}{81} + \frac{1}{16} \times \frac{x^3}{729} - \dots\right) \\ &= 1 + \frac{1}{6}x - \frac{1}{216}x^2 + \frac{1}{3888}x^3 - \dots \\ \sqrt{9.1} &= \sqrt{9+0.1} \approx 3 + \frac{1}{6} \times 0.1 - \frac{1}{216} \times 0.1^2 = 3.017 \end{aligned}$$

$$\begin{aligned} b \quad \frac{1}{(1+x)^4} &= (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n \\ &= 1 - 4x + 10x^2 - 20x^3 + 35x^4 - \dots \\ \frac{1}{1.03^4} &= \frac{1}{(1+0.03)^4} \\ &\approx 1 - 4 \times 0.03 + 10 \times 0.03^2 - 20 \times 0.03^3 = 0.8885 \end{aligned}$$

Exercises 5G

$$1 \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

When $x = -1$,

$$e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots$$

The series alternates and the last term is less than 0.001

Error $< \frac{1}{5040}$, which is approximately 0.0002, hence

$$e^{-1} \approx 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.36806.$$

$$2 \quad \ln(1+x) = \int \frac{1}{1+x} dx = \int \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c.$$

$$|R_n| < u_{n+1} \Rightarrow \frac{0.5^{n+1}}{n+1} < 0.0001 \Rightarrow n = 9, \text{ and}$$

$$\sum_{n=0}^9 (-1)^n \frac{(0.5)^{n+1}}{n+1} = 0.405435$$

$$3 \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \leq 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

$$= 1 + \frac{1}{1 - \frac{1}{2}} = 3. \text{ Hence, } e < 3.$$

Since $|R_n(1)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} = \frac{e^c}{(n+1)!}$, and e^x is a strictly increasing function,

$$e^c < e < 1 \Rightarrow |R_n(1)| < \frac{3}{(n+1)!} < 0.00005 \Rightarrow n \geq 8.$$

$$\text{Hence, } e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} = 2.7183.$$

$$4 \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_5(x);$$

$$R_5(x) = \frac{f^{(6)}(c)x^6}{6!} = \frac{-\sin c \times x^6}{6!}; c \in \left[-\frac{1}{3}, \frac{1}{3}\right].$$

$f^{(6)}(c)$ is maximized when $c = \frac{1}{3}$.
Hence, the maximum error is

$$\left(\frac{1}{3}\right)^6 \times \sin\left(\frac{1}{3}\right) \approx 6.2337 \times 10^{-7}.$$

$$5 \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$\text{For } x = \frac{1}{3}, R_n\left(\frac{1}{3}\right) = \frac{f^{(n+1)}(c)\left(\frac{1}{3}\right)^{n+1}}{(n+1)!}, 0 < c < \frac{1}{3}.$$

$$\text{Hence, } \left|R_n\left(\frac{1}{3}\right)\right| < \frac{e^{\frac{1}{3}}}{(n+1)!} \times \left(\frac{1}{3}\right)^{n+1}. \text{ Since } e < 27, e^{\frac{1}{3}} < 3,$$

and we need

$$\left|R_n\left(\frac{1}{3}\right)\right| < 0.00005 \Rightarrow \frac{1}{(n+1)! \times 3^n} < 0.00005.$$

$$\text{Therefore, } n = 5 \text{ since } \frac{1}{(5+1)! \times 3^5} \approx 0.000006 < 0.00005.$$

Hence,

$$T_5\left(\frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)^3}{3!} + \frac{\left(\frac{1}{3}\right)^4}{4!} + \frac{\left(\frac{1}{3}\right)^5}{5!} \approx 1.3956.$$

$$6 \quad f(x) = x^{\frac{1}{3}}; f'(x) = \frac{1}{3}x^{-\frac{2}{3}};$$

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}; f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}.$$

$$f(8) = 2; f'(8) = \frac{1}{12}; f''(8) = -\frac{1}{144}.$$

$$\text{Hence, } T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

and $x^{\frac{1}{3}} \approx T_2(x)$.

$$|R_2(x)| = \frac{f'''(c)}{3!}(x-8)^3 = \frac{10}{27}c^{-\frac{8}{3}} \frac{(x-8)^3}{3!} = \frac{5(x-8)^3}{81c^{\frac{8}{3}}},$$

where $8 < c < x$. Since $7 \leq x \leq 9$, $-1 \leq (x-8)$, or $|x-8| \leq 1$, and therefore

$$|x-8|^3 \leq 1. x > 7 \Rightarrow c^{\frac{8}{3}} > 7^{\frac{8}{3}} > 179.$$

$$\text{Hence, } |R_2(x)| = \frac{5|x-8|^3}{81c^3} < \frac{5 \times 1}{81 \times 179} < 0.0004, \text{ i.e.,}$$

the approximation is accurate to within 0.0004.

$$7 \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \Rightarrow \sin^2 x$$

$$= (\sin x)(\sin x) = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots$$

$$\int_0^{\frac{\pi}{6}} \left(x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{1}{15}x^5 + \frac{2}{315}x^7 \right]_0^{\frac{\pi}{6}} = 0.045294\dots$$

$$\text{The exact integral is, } \int_0^{\frac{\pi}{6}} \sin^2 x dx = \int_0^{\frac{\pi}{6}} \frac{1 - 2\cos x}{2} dx$$

$$= \left[\frac{x}{2} - \frac{\sin \frac{\pi}{3}}{4} \right]_0^{\frac{\pi}{6}} = 0.045293\dots$$

$$\text{The truncated term } |R_4(x)| = \frac{8}{315}x^8,$$

$$\int_0^{\frac{\pi}{6}} \frac{x^8}{315} dx = \left[\frac{x^9}{2835} \right]_0^{\frac{\pi}{6}} = 0.000001. \text{ The actual}$$

error is $0.0452940 - 0.045293 = 0.000001$ and

$$|R_4(x)| \leq \text{actual error.}$$

$$8 \quad R_5\left(\frac{1}{2}\right) = \frac{f^{(6)}(c)}{6!} \times \left(\frac{1}{2}\right)^6, 0 < c < \frac{1}{2}, \text{ and } |f^{(6)}(c)| < 1.$$

$$|R_5(x)| \leq \frac{1}{6! \times 26} \approx 0.0000217.$$

$$9 \quad R_4(0.2) = \frac{f^{(5)}(c)}{5!} \times (0.2)^5; 0 < c < 0.2. \text{ Since}$$

$f^{(5)}(x) = e^x$, the maximum that

$f^{(5)}(c)$ can be in this interval is $e^{0.2}$,

$$\text{hence } |R_4(0.2)| \leq \frac{e^{0.2}}{5!} \times (0.2)^5 \approx 0.0000326.$$

$$10 \quad |R_n(1)| \leq \left| \frac{\sin^{(n)}(1)}{(n+1)!} \right|. \text{ Since all derivatives of } \sin(x) \text{ are}$$

between -1 and 1 , $|R_n(1)| \leq \frac{1}{(n+1)!}$. By trial and

$$\text{error, } n = 6, \frac{1}{7!} = \frac{1}{5050} < 0.001$$

The degree of the polynomial is therefore 5.

Exercise 5H

$$1 \quad a \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) - x}{x^3}$$

$$= -\frac{1}{3!} \lim_{x \rightarrow 0} \frac{x^2}{5!} + \frac{x^4}{7!} + \dots = -\frac{1}{6}$$

$$b \quad \lim_{x \rightarrow 0} \frac{\left(1 + x + \frac{x^2}{2!} + \dots \right) - \left(1 - x + \frac{x^2}{2!} - \dots \right)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{2x + \frac{2x^2}{2!} + \frac{2x^4}{4!} \dots}{x} = 2 \lim_{x \rightarrow 0} \frac{2x}{2!} + \frac{2x^3}{4!} + \dots = 2$$

$$c \quad \lim_{x \rightarrow 0} \frac{(2 - 2\cos x)^3}{x^6} = 8 \lim_{x \rightarrow 0} \frac{(1 - \cos x)^3}{x^6}$$

$$= 8 \lim_{x \rightarrow 0} \frac{1 - 3\cos x + 3\cos^2 x - \cos^3 x}{x^6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos^2 x = (\cos x)(\cos x)$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \dots$$

$$\cos x = (\cos x)(\cos^2 x) = 1 - \frac{3x^2}{2} + \frac{7x^4}{8} - \frac{61x^6}{240} + \dots$$

$$8 \lim_{x \rightarrow 0} \frac{1 - 3\cos x + 3\cos^2 x - \cos^3 x}{x^6} = 8 \lim_{x \rightarrow 0} \frac{1}{8} x^6 = 1$$

$$d \quad \sin^2 x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \dots$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^4 - \frac{2}{45}x^6 + \dots}{x^4 - \frac{x^6}{3} + \dots} = \frac{1}{3}.$$

$$e \quad \lim_{x \rightarrow 0} \frac{(\sin 3x - 3x)^2}{(\cos 5x - 1)^3}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 3x + 3x \sin 3x + 9x^2}{\cos^3 5x - 3\cos^2 5x + 3\cos 5x - 1} = -\frac{162}{15625}$$

$$\text{Numerator: } \left[(3x)^2 - \frac{(3x)^4}{3} + \frac{2(3x)^6}{45} - \dots \right]$$

$$- 3x \left[3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120} - \dots \right] + (3x)^2$$

Denominator:

$$\left[1 - \frac{3(5x)^2}{2} + \frac{7(5x)^4}{8} - \dots \right] - 3 \left[1 - (5x)^2 + \frac{(5x)^4}{3} - \dots \right]$$

$$+ 3 \left[1 - \frac{(5x)^2}{2} + \frac{(5x)^4}{24} - \dots \right] - 1$$

2 For $c = 0$, $y = y(0) + y'(0) + \frac{y''(0)}{2!} + \frac{y'''(0)}{3!} + \dots$

Therefore, $y(0) = 0$;

$$y' = y^2 - x; y'(0) = 1$$

$$y'' = 2yy' - 1; y''(0) = 1$$

$$y''' = 2yy'' + 2(y')^2; y'''(0) = 4$$

$$y^{(4)} = 2yy''' + 6y'y''; y^{(4)}(0) = 14$$

$$y^{(5)} = 2yy^{(4)} + 8y'y''' + 6(y'')^2; y^{(5)}(0) = 66$$

$$\text{Hence, } y = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \dots$$

$$y(0.2) = 1.2264 \text{ (4d.p.)}$$

3 $y' = 3x^2y$;

$$y'' = 3x^2y' + 6xy = 3x^2(3x^2y) + 6xy = 6x^4y + 9x^4y$$

$$y''' = 6y + 6xy' + 36x^3y + 9x^4y'$$

$$= 6y + 6x(3x^2y) + 36x^3y + 9x^4(3x^2y)$$

$$= 6y + 54x^3y + 27x^6y$$

$$\text{For } (1, 1), y' = 3; y'' = 15; y''' = 87$$

Therefore, using the Taylor series expansion,

$$y = 1 + (x-1)(3) + \frac{(x-1)^2}{2!}(15) + \frac{(x-1)^3}{3!}(87)$$

$$= 1 + 3(x-1) + 7.5(x-1)^2 + 14.5(x-1)^3$$

- 4 In this example we will purposely take a different approach to show you another way of solving this type of problem.

$$\text{Let } y = \sum c_n x^n; y' = \sum n c_n x^{n-1}.$$

$$\text{Since } x^2 y' = y - x - 1,$$

$$x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = -1 - x + \sum_{n=1}^{\infty} c_n x^n, \text{ so that}$$

$$\sum_{n=1}^{\infty} n c_n x^{n+1} = -1 - x + \sum_{n=0}^{\infty} c_n x^n. \text{ Moving the}$$

summation index on the left by -1 , i.e., replace $n = 1$ with $n = 2$ and n with $(n-1)$ we obtain

$$\sum_{n=2}^{\infty} (n-1) c_{n-1} x^n = -1 - x + c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n.$$

$$c_0 = 1 \times c_1 = 1!; c_3 = 2 \times c_2 = 2!;$$

$$c_4 = 3 \times c_3 = 3!; \dots; c_n = (n-1)!, n \geq 2.$$

$$\text{Hence, } y(x) = 1 + x + \sum_{n=2}^{\infty} (n-1)! x^n.$$

$$\text{The radius of convergence is } \lim_{n \rightarrow \infty} \frac{(n-1)!}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Therefore the series converges only for $x = 0$.

Hence, the differential equation does not have a convergent power series solution.

Review Exercise

- 1 Using the ratio test,

$$\lim_{n \rightarrow \infty} \left(\frac{(2n)! x^{n+1}}{n!(n+1)!} \times \frac{n!(n-1)!}{(2n-2)! x^n} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2n(2n-1)}{n(n+1)} \right) x = 4x. \text{ The series is convergent}$$

$$\text{if } |4x| < 1, \text{ or } x \in \left[-\frac{1}{4}, \frac{1}{4}\right], \text{ hence } R = \frac{1}{4}.$$

2 a i $f'(x) = -e^{-x} \cos 2x - 2e^{-x} \sin 2x$

$$f''(x) = 4e^{-x} \sin 2x - 3e^{-x} \cos 2x$$

$$f''(x) + 2f'(x) + 5f(x)$$

$$= 4e^{-x} \sin 2x - 3e^{-x} \cos 2x - 2e^{-x} \cos 2x - 4e^{-x} \sin 2x + 5e^{-x} \cos 2x.$$

$$\text{Hence, } f''(x) + 2f'(x) + 5f(x) = 0.$$

- ii Differentiating n times and putting $x = 0$,

$$f^{(n+2)}(0) + 2f^{(n+1)}(0) + 5f^{(n)}(0) = 0.$$

b $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \dots$ and

$$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} + \dots$$

Multiplying these,

$$e^{-x} \cos 2x = 1 - x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{7}{24}x^4 + \dots$$

- 3 a f is a continuous and positive decreasing function, hence we can use the integral test.

$$\int_1^{\infty} \frac{x}{e^{x^2}} dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-x^2} \right]_1^b = \frac{1}{2e}.$$

Since the integral converges, the series converges.

b $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\text{Hence, } e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

c $e^{x^2} \sin x = \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots \right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$

$$= x + \frac{5}{6}x^3 + \frac{41}{120}x^5 + \dots$$

d $\frac{e^{x^2} \sin x - x}{x^3} = \frac{5}{6} + \frac{41}{120}x^2.$

$$\text{Hence, } \lim_{x \rightarrow 0} \left(\frac{e^{x^2} \sin x - x}{x^3} \right) = \frac{5}{6}.$$

4 a i $a_n = \frac{f^{(n)}(0)}{n!}$ or $f^{(n)}(0) = n!a_n$.

When $x = 0$, $(n+2)!a_{n+2} - n^2 \times n!a_n = 0$.

Hence, $(n+1)(n+2)a_{n+2} = n^2a_n$, $n \geq 1$.

ii $a_3 = \frac{1^2}{2 \times 3}$; $a_5 = \frac{1^2}{2 \times 3} \times \frac{3^2}{4 \times 5}$.

Hence, $a_n = \frac{1^2 \times 3^2 \dots (n-2)^2}{n!}$, $n \in \text{Odds}$, $n \geq 3$.

b Using the ratio test, for odd n ,

$\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)(n+2)} x^2 = x^2$. Hence, the series is

convergent for $|x| < 1$, and $R = 1$.

c $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \approx 1 \times \frac{1}{2} + \frac{1}{6} \times \left(\frac{1}{2}\right)^3 + \frac{3}{40}\left(\frac{1}{2}\right)^5$
 ≈ 3.139 .

5 $\lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{2^{n+1}\sqrt{n+1}} \times \frac{2^n\sqrt{n}}{(x-3)^n} \right| = \frac{|x-3|}{2}$. The series will

converge for $\frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$.

Checking the endpoints, $x = 1$, $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

This converges by the alternating series test, since the terms decrease to zero. For $x = 5$,

$\sum_{n=1}^{\infty} \frac{(2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the p-series test.

Hence, the interval of convergence is $[1, 5]$

6 Maclaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $|R_n(x)| \leq \frac{M}{(n+1)!} x^{n+1}$.

$|R_n(0.2)| \leq \frac{e^{0.2}}{(n+1)!} 0.2^{n+1} < 0.0005$.

$e < 3 \Rightarrow e^{0.2} < e^{0.5} < 2$

Then, $|R_n(0.2)| \leq \frac{2}{(n+1)!} 0.2^{n+1} < 0.0005 \Rightarrow n = 3$

Hence, $e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2!} + \frac{0.2^3}{3!} = 1.221$

7 $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + c$

Since $\ln(1) = 0$, $c = 0$.

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$,

hence $\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2} (\ln(1+x) - \ln(1-x))$

$= \frac{1}{2} \left(\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \right) \right)$

$= \frac{1}{2} \left(2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots \right)$

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$

8 $f(x) = (1+x)^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$;

$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$; $f'''(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$.

Hence, $f(x) \approx \sum_{i=0}^3 \frac{f^{(i)}(0)}{i!} x^i = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$;

$f(0.2) \approx 1.06272$.

9 $3^\circ = \frac{\pi}{180} \times 3 = \frac{\pi}{60}$.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + R_n$

$\Rightarrow \sin x = 0 + x - \frac{x^3}{3!} + \dots + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$

$f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, then $f^{(n+1)}(c) = 1$.

$|R_n| \leq \frac{\left| \frac{\pi}{60} \right|^{n+1}}{(n+1)!} \leq 0.000005 \Rightarrow n = 3$.

$\sin 3^\circ \approx \frac{\pi}{60} - \frac{\left(\frac{\pi}{60} \right)^3}{3!} \approx 0.05234$.

10 a $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$

b Coefficient of $x^n = \frac{f^{(n)}(0)}{n!} = \frac{1 + (-1)^n}{2n!}$

$$\Rightarrow f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

c $f\left(\frac{1}{2}\right) = 1 + \frac{1}{8} + \frac{1}{16 \times 24} = \frac{433}{384}$. The Lagrange

error term $\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{f^{(5)}(c)}{120} \times \left(\frac{1}{2}\right)^5$.

From the GDC, $f^{(5)}(c)$ is an increasing function, hence the expression is maximized when $x = 0.5$. Therefore the upper bound

$$= \frac{(e^{0.5} - e^{-0.5})}{2 \times 120} \times \left(\frac{1}{2}\right)^5 = 0.000136.$$

11 a $f(x) = \ln(1 + \sin x)$. Hence, $f'(x) = \frac{\cos x}{1 + \sin x}$;

$$f''(x) = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2}$$

$$= \frac{-\sin x - 1}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}.$$

b $f'''(x) = \frac{\cos x}{(1 + \sin x)^2}$; $f^{(4)}(x) = \frac{-\sin x(1 + \sin x)^2 - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}$.

$$f(0) = 0; f'(0) = 1; f''(0) = -1;$$

$$f'''(0) = 1; f^{(4)}(0) = -2.$$

Therefore, $\ln(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \dots$

c $\ln(1 - \sin x) = \ln(1 + \sin(-x))$

$$= -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \dots$$

d Adding,

$$\ln(1 - \sin^2 x) = \ln \cos^2 x = -x^2 - \frac{x^4}{6} + \dots$$

$$\ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} + \dots$$

$$\ln \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \dots$$

e $\frac{\ln \sec x}{x\sqrt{x}} = \frac{\sqrt{x}}{2} + \frac{x^2\sqrt{x}}{12} + \dots$

Hence, $\lim_{x \rightarrow 0} \frac{\ln \sec x}{x\sqrt{x}} = 0$.

12 $y(0) = 0$

$$y' = y^2 + 1; y'(0) = 1$$

$$y'' = 2yy'; y''(0) = 0$$

$$y''' = 2[(y')^2 + yy'']; y'''(0) = 2; y^{(5)}(0) = 16$$

$$y^{(4)} = 2[2y'y'' + y'y''']$$

$$= 2[3y'y'' + yy''']; y^{(4)}(0) = 0$$

$$y^{(5)} = 2[3((y'')^2 + y'y''') + y'y''' + yy^{(4)}]$$

$$= 2[3(y'')^2 + 4y'y''' + yy^{(4)}]$$

$$y = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \dots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots,$$

which is the Maclaurin expansion of

$$y = \tan(x).$$

13 a $y' = y \tan x + \cos x$, $f'(0) = 1$

$$y'' = y \sec^2 x + y' \tan x - \sin x$$

$$\Rightarrow f''(0) = -\frac{\pi}{2}$$

$$\Rightarrow y = -\frac{\pi}{2} + x - \frac{\pi x^2}{4}$$

b integrating factor

$$= e^{\int -\tan x dx} = \cos x$$

$$\Rightarrow y \cos x = \int \cos^2 x dx = \frac{1}{2} \int (1 + \cos 2x) dx$$

$$\Rightarrow y \cos x = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

when

$$x = \pi, y = 0 \Rightarrow c = -\frac{\pi}{2}$$

$$\Rightarrow y \cos x = \frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2}$$

$$\Rightarrow y = \sec x \left(\frac{x}{2} + \frac{\sin 2x}{4} - \frac{\pi}{2} \right)$$

$$14 \sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$= 1 + \left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \frac{x^2}{2!} + \left(\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \frac{x^3}{3!} + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

$$I =]-1, 1[\frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)(-2x) + \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \frac{(-2x)^2}{2!}$$

$$+ \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \frac{(-2x)^3}{3!} + \dots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots$$

$$I = \left] -\frac{1}{2}, \frac{1}{2} \right[$$

$$(1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}}$$

$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots\right) \left(1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots\right)$$

$$= 1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{51}{16}x^3 + \dots$$

$$I = \left] -\frac{1}{2}, \frac{1}{2} \right[$$

$$15 e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$e^{e^x-1} = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^2}{2}$$

$$+ \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^3}{6} + \dots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$= 1 + x + x^2 + \frac{5}{6}x^3 + \dots$$

Using l'Hopital's rule,

$$\lim_{x \rightarrow 0} \frac{e^{(e^x-1)} - 1}{e^{(e^x-1)} - 1 - 1} = \lim_{x \rightarrow 0} \frac{e^{(e^x-1)} - 1}{e^{(e^x+x-1)} - 1}$$

$$= \lim_{x \rightarrow 0} \frac{e^{(e^x+x-1)} - 1}{e^{(e^x+x-1)} \times (e^x + 1)} = \frac{1}{2}$$

$$16 \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \Rightarrow \pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4}{2n-1}$$

$$\frac{4}{2n-1} < 5 \times 10^{-5} \Rightarrow n \geq 4,001$$

$$17 e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \dots + \frac{\left(-\frac{x^2}{2}\right)^n}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n n!}$$

$$\text{Hence, } P(0 < x < 1) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{2^n n!} dx$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+1}}{(2n+1)2^n n!} \right]_0^1$$

Since this is an alternating series,

$$u_{n+1} = \frac{1}{(2n+3)2^{n+1}(n+1)!} < 0.0001.$$

From the GDC:

When $n = 4$, we have the appropriate approximation, i.e., 0.341354.